

Finite element schemes for a class of nonlocal parabolic systems with moving boundaries

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Abstract

The aim of this paper is to establish convergence, properties and error bounds for the fully discrete solutions of a class of nonlinear systems of reaction-diffusion nonlocal type with moving boundaries, using the finite element method with polynomial approximations of any degree. A coordinate transformation which fixes the boundaries is used. Some numerical tests to compare our Matlab code with a moving finite element method are investigated.

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1 Introduction

In this work, we study parabolic systems with nonlocal nonlinearity of the following type:

$$\begin{cases} \frac{\partial u_i}{\partial t} - a_i \left(\int_{\Omega_t} u_1(x, t) dx, \dots, \int_{\Omega_t} u_{n_e}(x, t) dx \right) \frac{\partial^2 u_i}{\partial x^2} = f_i(x, t), & (x, t) \in Q_t \\ u_i(\alpha(t), t) = u_i(\beta(t), t) = 0, & t > 0 \\ u_i(x, 0) = u_{i0}(x), & x \in \Omega_0 =]\alpha(0), \beta(0)[, \quad i = 1, \dots, n_e \end{cases} \quad (1)$$

where Q_t is a bounded non-cylindrical domain defined by

$$Q_t = \{(x, t) \in \mathbb{R}^2 : \alpha(t) < x < \beta(t), \text{ for all } 0 < t < T\}$$

and

$$\Omega_t = \{x \in \mathbb{R} : \alpha(t) < x < \beta(t), 0 \leq t \leq T\}.$$

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Problem (1) arises in a large class of real models, for example, in biology, where the solution u could describe the density of a population subject to spreading; or in physics, where u could represent the temperature, considering that the measurements are an average in a neighbourhood [9]. It is nonlocal in the sense that the diffusion coefficient is determined by a global quantity, that is, a depends on the whole population in the area.

This class of problems, with nonlocal coefficients in an open bounded cylindrical domain, was initially studied by Chipot and Lovat in [10], where they proved the existence and uniqueness of weak solutions. In recent years, nonlinear parabolic equations with nonlocal diffusion terms have been extensively studied [11, 1, 14, 8, 12, 13, 15, 25], especially in relation to questions of existence, uniqueness and asymptotic behaviour.

In order to model interactions, a system is needed. Raposo et al. [20], in 2008, studied the existence, uniqueness and exponential decay of solutions for reaction-diffusion coupled systems of the form

$$\begin{cases} u_t - a(l(u))\Delta u + f(u - v) = \alpha(u - v) & \text{in } \Omega \times]0, T], \\ v_t - a(l(v))\Delta v - f(u - v) = \alpha(v - u) & \text{in } \Omega \times]0, T], \end{cases}$$

with $a(\cdot) > 0$, l a continuous linear form, f a Lipschitz-continuous function and α a positive parameter. Recently, Duque et al. [16] considered nonlinear systems of parabolic equations with a more general nonlocal diffusion term working on two linear forms l_1 and l_2 :

$$\begin{cases} u_t - a_1(l_1(u), l_2(v))\Delta u + \lambda_1|u|^{p-2}u = f_1(x, t) & \text{in } \Omega \times]0, T], \\ v_t - a_2(l_1(u), l_2(v))\Delta v + \lambda_2|v|^{p-2}v = f_2(x, t) & \text{in } \Omega \times]0, T]. \end{cases} \quad (2)$$

They gave important results on polynomial and exponential decay, vanishing of the solutions in finite time, and localisation properties such as waiting time effect.

Moving boundary problems occur in many physical applications involving diffusion, such as in heat transfer where a phase transition occurs, in moisture transport such as swelling grains or polymers, and in deformable porous media problems where the solid displacement is governed by diffusion (see, for example, [19, 3, 22, 6, 5]). Cavalcanti et al [7] worked with a time-dependent function $a = a\left(t, \int_{\Omega_t} |\nabla u(x, t)|^2 dx\right)$ to establish the solvability and exponential energy decay of the solution for a model given by a hyperbolic-parabolic equation in an open bounded subset of \mathbb{R}^n , with moving boundary. Santos et al. [23] established the exponential energy decay of the solutions for nonlinear coupled systems of beam equations with memory in noncylindrical domains. Recently, Robalo et al. [21] proved the existence and uniqueness of weak and strong global in time solutions and gave conditions, on the data, for these solutions to have the exponential decay property. The analysis and numerical simulation of such problems presents further challenges. In [1], Ackleh and Ke propose a finite difference scheme to approximate the solutions and to study their long time behaviour. The authors also made numerical simulations, using an implicit finite difference scheme in one dimension [20] and the finite volume discretisation in

two space dimensions [18]. Bendahmane and Sepulveda [4], in 2009, investigated the propagation of an epidemic disease modelled by a system of three PDE, where the i th equation is of the type

$$(u_i)_t - a_i \left(\int_{\Omega} u_i dx \right) \Delta u_i = f_i(u_1, u_2, u_3),$$

in a physical domain $\Omega \subset \mathbb{R}^n$ ($n = 1, 2, 3$). They established the existence of solutions for finite volume schemes and their convergence to the weak solution of the PDE. In [17], the authors proved the optimal order of convergence for a linearised Euler-Galerkin finite element method for problem (2) and presented some numerical results. Almeida et al., in [2], established convergence, properties and error bounds for the fully discrete solutions of a class of non-linear equations of reaction-diffusion nonlocal type with moving boundaries, using a linearised Crank-Nicolson-Galerkin finite element method with polynomial approximations of arbitrary degree. In [21], Robalo et al. also obtained approximate numerical solutions for equations of this type with a Matlab code based on the Moving Finite Element Method (MFEM) with high degree local approximations.

In this paper, we study the convergence of the total discrete solutions using the finite element method with some classical time integrators. To the best of our knowledge, these results are new for nonlocal reaction-diffusion systems with moving boundaries.

The paper is organized as follows. In Section 2, we formulate the problem and the hypotheses on the data. In Section 3, we define and prove the convergence of the semidiscrete solution. Section 4 is devoted to the proof of the existence, uniqueness, stability and convergence of the fully discrete solutions for each method. In Section 5, we obtain and compare the approximate numerical solutions for one example. Finally, in Section 6, we draw some conclusions.

2 Statement of the problem

In what follows, we study the convergence of the totally discrete solutions of the one-dimensional Dirichlet problem with two moving boundaries, defined by

$$\begin{cases} \frac{\partial u_i}{\partial t} - a_i \left(\int_{\Omega_t} u_1(x, t) dx, \dots, \int_{\Omega_t} u_{n_e}(x, t) dx \right) \frac{\partial^2 u_i}{\partial x^2} = f_i(x, t), & (x, t) \in Q_t \\ u_i(\alpha(t), t) = u_i(\beta(t), t) = 0, & t > 0 \\ u_i(x, 0) = u_{i0}(x), & x \in \Omega_0 =]\alpha(0), \beta(0)[, \quad i = 1, \dots, n_e \end{cases} \quad (3)$$

where

$$Q_t = \{(x, t) \in \mathbb{R}^2 : \alpha(t) < x < \beta(t), \text{ for all } 0 < t < T\}$$

is a bounded non-cylindrical domain, T is an arbitrary positive real number, a_i denotes a positive real function and

$$\Omega_t = \{x \in \mathbb{R} : \alpha(t) < x < \beta(t), 0 \leq t \leq T\}.$$

The lateral boundary of Q_t is given by $\Sigma_t = \bigcup_{0 \leq t < T} (\{\alpha(t), \beta(t)\} \times \{t\})$.

In [21], the authors established the existence, uniqueness and asymptotic behaviour of strong regular solutions for these type of problems using a coordinate transformation, which fixes the boundaries, and assuming that the real function $\gamma(t) = \beta(t) - \alpha(t)$ is increasing on $0 \leq t < T$. They used the fact that, when (x, t) varies in Q_t , the point (y, t) of \mathbb{R}^2 , with $y = (x - \alpha(t))/\gamma(t)$, varies in the cylinder $Q =]0, 1[\times]0, T[$. Thus, the function $\tau : Q_t \rightarrow Q$ given by $\tau(x, t) = (y, t)$, is of class \mathcal{C}^2 . The inverse τ^{-1} is also of class \mathcal{C}^2 . The change of variable $v(y, t) = u(x, t)$ and $g(y, t) = f(x, t)$ with $x = \alpha(t) + \gamma(t)y$ transforms problem (3) into the following problem:

$$\begin{cases} \frac{\partial v_i}{\partial t} - a_i(l(v_1), \dots, l(v_{n_e})) b_2(t) \frac{\partial^2 v_i}{\partial y^2} - b_1(y, t) \frac{\partial v_i}{\partial y} = g_i(y, t), & (y, t) \in Q \\ v_i(0, t) = v_i(1, t) = 0, & t > 0 \\ v_i(y, 0) = v_{i0}(y), & y \in \Omega =]0, 1[, \quad i = 1, \dots, n_e \end{cases} \quad (4)$$

where $l(v) = \gamma(t) \int_0^1 v(y, t) dy$, $g_i(y, t) = f_i(\alpha + \gamma y, t)$ and $v_{i0}(y) = u_{i0}(\alpha(0) + \gamma(0)y)$. The coefficients $b_1(y, t)$ and $b_2(t)$ are defined by

$$b_1(y, t) = \frac{\alpha'(t) + \gamma'(t)y}{\gamma(t)} \quad \text{and} \quad b_2(t) = \frac{1}{(\gamma(t))^2}.$$

With this change of variable, we transfer the problem of the boundary's movement to the first order advection term. If the speed of the boundary grows fast with time, b_1 can dominate in magnitude the diffusion coefficient, which can result in numerical instability. Thus, some conditions must be imposed on the mesh size and on the time step. We will address this issue later.

Since we need the existence and uniqueness of a strong solution in Q_t , we will assume that the hypotheses in [21] are satisfied, namely:

- (H1) $\alpha, \beta \in \mathcal{C}^2([0, T])$ and $0 < \gamma_0 < \gamma(t) < \gamma_1 < \infty$, for all $t \in [0, T]$
- (H2) $\alpha', \beta' \in L_2(]0, T[)$
- (H3) $u_{i0} \in H_0^1(\Omega_0)$, $\Omega_0 =]\alpha(0), \beta(0)[$, $i = 1, \dots, n_e$,
- (H4) $\int_0^T \int_{\Omega_t} f_i^2 dx dt < \infty$, $\Omega_t =]\alpha(t), \beta(t)[$, $i = 1, \dots, n_e$,
- (H5) $a_i : \mathbb{R}^{n_e} \rightarrow \mathbb{R}^+$ is Lipschitz-continuous
with $0 < m_a \leq a_i(s) \leq M_a$, for all $s \in \mathbb{R}$, $i = 1, \dots, n_e$.

We also need to assume that

$$|\gamma'(t)| \leq \gamma'_{\max} \text{ and } |\alpha'(t)| \leq \alpha'_{\max}.$$

Let $\Omega =]0, 1[$. The definition of a weak solution is as follows.

Definition 1 (Weak solution). *We say that the function $\mathbf{v} = (v_1, \dots, v_{n_e})$ is a weak solution of problem (4) if, for each $i \in \{1, \dots, n_e\}$,*

$$v_i \in L_\infty(0, T; H_0^1(\Omega) \cap H^2(\Omega)), \frac{\partial v_i}{\partial t} \in L_2(0, T; L_2(\Omega)), \quad (5)$$

the following equality is valid for all $w_i \in H_0^1(\Omega)$, and $t \in]0, T[$,

$$\int_0^1 \frac{\partial v_i}{\partial t} w_i dy + a_i(l(v_1), \dots, l(v_{n_e})) b_2 \int_0^1 \frac{\partial v_i}{\partial y} \frac{\partial w_i}{\partial y} dy - \int_0^1 b_1 \frac{\partial v_i}{\partial y} w_i dy = \int_0^1 g_i w_i dy \quad (6)$$

and

$$v_i(x, 0) = v_{i0}(x), \quad x \in \Omega \quad (7)$$

3 Semidiscrete solution

We denote the usual L_2 norm and inner product in Ω by $\|\cdot\|$ and (\cdot, \cdot) respectively, and the norm in $H^k(\Omega)$ by $\|\cdot\|_{H^k}$. Let \mathcal{T}_h denote a partition of Ω into disjoint intervals T_i , $i = 1, \dots, n_t$, such that $h = \max\{\text{diam}(T_i), i = 1, \dots, n_t\}$. Now, let S_h^k denote the continuous functions on the closure $\bar{\Omega}$ of Ω which are polynomials of degree k in each interval of \mathcal{T}_h and which vanish on $\partial\Omega$, that is,

$$S_h^k = \{W \in C_0^0(\bar{\Omega}) | W|_{T_i} \text{ is a polynomial of degree } k \text{ for all } T_i \in \mathcal{T}_h\}.$$

If $\{\varphi_j\}_{j=1}^{n_p}$ is the Lagrange basis for S_h^k associated to the points $\{P_j\}_{j=1}^{n_p}$, then we can represent each $W \in S_h^k$ as

$$W = \sum_{j=1}^{n_p} W(P_j) \varphi_j.$$

Given a smooth function u on Ω , which vanishes on $\partial\Omega$, we may define its interpolant, denoted by $I_h u$, as the function of S_h^k which coincides with u at the points $\{P_j\}_{j=1}^{n_p}$, that is,

$$I_h u = \sum_{j=1}^{n_p} u(P_j) \varphi_j.$$

Lemma 2 ([24]). *If $u \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$, then*

$$\|I_h u - u\| + h \|\nabla(I_h u - u)\| \leq C h^{k+1} \|u\|_{H^{k+1}}.$$

Definition 3 ([24]). *A function $\tilde{u} \in S_h^k$ is said to be the Ritz projection of $u \in H_0^1(\Omega)$ onto S_h^k if it satisfies*

$$(\nabla \tilde{u}, \nabla W) = (\nabla u, \nabla W), \quad \text{for all } W \in S_h^k.$$

Lemma 4 ([24]). *If $u \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$, then*

$$\|\tilde{u} - u\| + h \|\nabla(\tilde{u} - u)\| \leq C h^{k+1} \|u\|_{H^{k+1}},$$

where C does not depend on h nor on k .

The semidiscrete problem, based on Definition 1, consists in finding $\mathbf{V} = (V_1, \dots, V_{n_e}) \in (S_h^k)^{n_e}$, for $t \geq 0$, such that for all $\mathbf{W} = (W_1, \dots, W_{n_e}) \in (S_h^k)^{n_e}$ and $t \in]0, T[$:

$$\begin{cases} \left(\frac{\partial V_i}{\partial t}, W_i \right) + a_i(l(V_1), \dots, l(V_{n_e})) b_2 \left(\frac{\partial V_i}{\partial y}, \frac{\partial W_i}{\partial y} \right) - \left(b_1 \frac{\partial V_i}{\partial y}, W_i \right) = (g_i, W_i) \\ V_i(y, 0) = I_h v_{i0}, \quad i = 1, \dots, n_e \end{cases} \quad (8)$$

Since the functions a_i are continuous, Caratheodory's Theorem implies the existence of a solution to system (8), and arguing as in the proof of Theorem 3 in [21], we can prove the uniqueness of this solution. In virtue of condition (H5), the convergence of the semidiscrete solution to the weak solution of problem (4) can be obtained using standard arguments, and hence we will only present the main steps of the proof and specify the dependence on the regularity of the weak solution.

Theorem 5. *If \mathbf{v} is the solution of problem (4) and \mathbf{V} is the solution of problem (8), then*

$$\|V_i - v_i\| \leq Ch^{k+1}, \quad t \in]0, T], \quad i = 1, \dots, n_e$$

where C may depend on $\sum_{i=1}^{n_e} \|v_{i0}\|_{H^{k+1}}$, $\sum_{i=1}^{n_e} \|v_i\|_{L_\infty(0, T; H^{k+1}(\Omega))}$, $\sum_{i=1}^{n_e} \left\| \frac{\partial v_i}{\partial y} \right\|_{L_\infty(0, T; L_2(\Omega))}$, $\sum_{i=1}^{n_e} \|v_i\|_{L_2(0, T; H^{k+1}(\Omega))}$ and $\sum_{i=1}^{n_e} \left\| \frac{\partial v_i}{\partial t} \right\|_{L_2(0, T; H^{k+1}(\Omega))}$, but does not depend on h , k or i .

Proof. Let $e_i = V_i - v_i$ be written as

$$e_i(y, t) = (V_i(y, t) - \tilde{V}_i(y, t)) + (\tilde{V}_i(y, t) - v_i(y, t)) = \theta_i(y, t) + \rho_i(y, t),$$

with $\tilde{V}_i(y, t) \in S_h^k$ being the Ritz projection of v_i . Then

$$\|e_i(y, t)\| \leq \|\theta_i(y, t)\| + \|\rho_i(y, t)\|$$

and, by Lemma 4, it follows that

$$\|\rho_i(y, t)\| \leq Ch^{k+1} \|v_i\|_{H^{k+1}}, \quad t \in [0, T].$$

Concerning $\|\theta_i(y, t)\|$, if

$$a_i^{(h)} = a_i(l(V_1), \dots, l(V_{n_e})),$$

then, for every $i \in \{1, \dots, n_e\}$, we have that

$$\begin{aligned}
& \left(\frac{\partial \theta_i}{\partial t}, W_i \right) + a_i^{(h)} b_2 \left(\frac{\partial \theta_i}{\partial y}, \frac{\partial W_i}{\partial y} \right) - \left(b_1 \frac{\partial \theta_i}{\partial y}, W_i \right) \\
&= \left(\frac{\partial V_i}{\partial t}, W_i \right) + a_i^{(h)} b_2 \left(\frac{\partial V_i}{\partial y}, \frac{\partial W_i}{\partial y} \right) - \left(b_1 \frac{\partial V_i}{\partial y}, W_i \right) \\
&\quad - \left(\frac{\partial \tilde{V}_i}{\partial t}, W_i \right) - a_i^{(h)} b_2 \left(\frac{\partial \tilde{V}_i}{\partial y}, \frac{\partial W_i}{\partial y} \right) + \left(b_1 \frac{\partial \tilde{V}_i}{\partial y}, W_i \right) \\
&= (g_i, W_i) - \left(\frac{\partial v_i}{\partial t}, W_i \right) - a_i b_2 \left(\frac{\partial \tilde{V}_i}{\partial y}, \frac{\partial W_i}{\partial y} \right) + \left(b_1 \frac{\partial v_i}{\partial y}, W_i \right) \\
&\quad + (a_i - a_i^{(h)}) b_2 \left(\frac{\partial \tilde{V}_i}{\partial y}, \frac{\partial W_i}{\partial y} \right) + \left(b_1 \left(\frac{\partial \tilde{V}_i}{\partial y} - \frac{\partial v_i}{\partial y} \right), W_i \right) \\
&\quad + \left(\frac{\partial v_i}{\partial t} - \frac{\partial \tilde{V}_i}{\partial t}, W_i \right) \\
&= (a_i - a_i^{(h)}) b_2 \left(\frac{\partial \tilde{V}_i}{\partial y}, \frac{\partial W_i}{\partial y} \right) + \left(b_1 \left(\frac{\partial \tilde{V}_i}{\partial y} - \frac{\partial v_i}{\partial y} \right), W_i \right) \\
&\quad + \left(\frac{\partial v_i}{\partial t} - \frac{\partial \tilde{V}_i}{\partial t}, W_i \right).
\end{aligned}$$

If we consider $W_i = \theta_i$, then

$$\begin{aligned}
\left(\frac{\partial \theta_i}{\partial t}, \theta_i \right) + a_i^{(h)} b_2 \left\| \frac{\partial \theta_i}{\partial y} \right\|^2 &= (a_i - a_i^{(h)}) b_2 \left(\frac{\partial \tilde{V}_i}{\partial y}, \frac{\partial \theta_i}{\partial y} \right) + \left(b_1 \frac{\partial \rho_i}{\partial y}, \theta_i \right) - \left(\frac{\partial \rho_i}{\partial t}, \theta_i \right) \\
&\quad + \left(b_1 \frac{\partial \theta_i}{\partial y}, \theta_i \right).
\end{aligned}$$

Integrating by parts the second and the fourth terms on the right side of the above equation, we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\theta_i\|^2 + a_i^{(h)} b_2 \left\| \frac{\partial \theta_i}{\partial y} \right\|^2 \\
&= (a_i - a_i^{(h)}) b_2 \left(\frac{\partial \tilde{V}_i}{\partial y}, \frac{\partial \theta_i}{\partial y} \right) - \left(\frac{\partial \rho_i}{\partial t}, \theta_i \right) - \frac{\gamma'(t)}{\gamma(t)} (\rho_i, \theta_i) - \left(b_1 \rho_i, \frac{\partial \theta_i}{\partial y} \right) - \\
&\quad \frac{\gamma'(t)}{2\gamma(t)} (\theta_i, \theta_i).
\end{aligned}$$

Taking the absolute value of the expression on the right hand side of this equation and considering the lower limits of a_i and b_i , it follows that

$$\frac{1}{2} \frac{d}{dt} \|\theta_i\|^2 + \frac{m_a}{\gamma_1^2} \left\| \frac{\partial \theta_i}{\partial y} \right\|^2$$

$$\begin{aligned}
&\leq \left| a_i - a_i^{(h)} \right| \frac{1}{\gamma_0^2} \int_0^1 \left| \frac{\partial \tilde{V}_i}{\partial y} \right| \left| \frac{\partial \theta_i}{\partial y} \right| dy + \int_0^1 \left| \frac{\partial \rho_i}{\partial t} \right| |\theta_i| dy + \frac{\gamma'_{\max}}{\gamma_0} \int_0^1 |\rho_i| |\theta_i| dy \\
&\quad + \frac{\alpha'_{\max} + \gamma'_{\max}}{\gamma_0} \int_0^1 |\rho_i| \left| \frac{\partial \theta_i}{\partial y} \right| dy + \frac{\gamma'_{\max}}{2\gamma_0} \int_0^1 |\theta_i|^2 dy \\
&\leq C_1 \left| a_i - a_i^{(h)} \right|^2 + \frac{m_a}{2\gamma_1^2} \int_0^1 \left| \frac{\partial \theta_i}{\partial y} \right|^2 dy + \frac{1}{2} \int_0^1 \left| \frac{\partial \rho_i}{\partial t} \right|^2 dy + \frac{1}{2} \int_0^1 |\theta_i|^2 dy \\
&\quad + \frac{\gamma'_{\max}}{2\gamma_0} \int_0^1 |\rho_i|^2 dy + \frac{\gamma'_{\max}}{2\gamma_0} \int_0^1 |\theta_i|^2 dy + C_2 \int_0^1 |\rho_i|^2 dy + \frac{m_a}{2\gamma_1^2} \int_0^1 \left| \frac{\partial \theta_i}{\partial y} \right|^2 dy,
\end{aligned}$$

with $C_1 = C_1(m_a, \gamma_0, \left\| \frac{\partial \tilde{V}_i}{\partial y} \right\|_{L_\infty(0,T;L_2(\Omega))})$.

Since $\left\| \frac{\partial \tilde{V}_i}{\partial y} \right\| \leq \left\| \frac{\partial v_i}{\partial y} \right\|$, we have that $C_1 = C_1(m_a, \gamma_0, \left\| \frac{\partial v_i}{\partial y} \right\|_{L_\infty(0,T;L_2(\Omega))})$. Then, by (H5),

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\theta_i\|^2 &\leq C_3 \sum_{j=1}^{n_e} \|\rho_j\|^2 + C_4 \sum_{j=1}^{n_e} \|\theta_j\|^2 + \frac{1}{2} \left\| \frac{\partial \rho_i}{\partial t} \right\|^2 + \frac{1}{2} \|\theta_i\|^2 \\
&\quad + \frac{\gamma'_{\max}}{2\gamma_0} \|\rho_i\|^2 + \frac{\gamma'_{\max}}{2\gamma_0} \|\theta_i\|^2 + C_2 \|\rho_i\|^2 \\
&\leq C \sum_{j=1}^{n_e} \|\theta_j\|^2 + C \sum_{j=1}^{n_e} \|\rho_j\|^2 + \frac{1}{2} \left\| \frac{\partial \rho_i}{\partial t} \right\|^2.
\end{aligned}$$

and now $C = C(m_a, \gamma_0, \gamma'_{\max}, \left\| \frac{\partial v_i}{\partial y} \right\|_{L_\infty(0,T;L_2(\Omega))})$. Hence, we obtain

$$\frac{d}{dt} \left(\sum_{i=1}^{n_e} \|\theta_i\|^2 \right) \leq C \sum_{i=1}^{n_e} \|\theta_i\|^2 + C \sum_{i=1}^{n_e} \|\rho_i\|^2 + \sum_{i=1}^{n_e} \left\| \frac{\partial \rho_i}{\partial t} \right\|^2.$$

Applying Gronwall's Theorem, we arrive at the inequality

$$\sum_{i=1}^{n_e} \|\theta_i\|^2 \leq C \sum_{i=1}^{n_e} \|\theta_i(y, 0)\|^2 + C \sum_{i=1}^{n_e} \int_0^T \|\rho_i\|^2 + \left\| \frac{\partial \rho_i}{\partial t} \right\|^2 dt.$$

By the hypothesis of the theorem, we have, for every $i \in \{1, \dots, n_e\}$,

$$\begin{aligned}
\|\theta_i(y, 0)\|^2 &\leq \|e_i(y, 0)\|^2 = \|V_i(y, 0) - v_{i0}\|^2 \leq Ch^{2(k+1)} \|v_{i0}\|_{H^{k+1}}^2, \\
\int_0^T \|\rho_i\|^2 dt &\leq CTh^{2(k+1)} \|v_i\|_{L_2(0,T;H^{k+1}(\Omega))}^2, \\
\int_0^T \left\| \frac{\partial \rho_i}{\partial t} \right\|^2 dt &\leq CTh^{2(k+1)} \left\| \frac{\partial v_i}{\partial t} \right\|_{L_2(0,T;H^{k+1}(\Omega))}^2
\end{aligned}$$

and so

$$\sum_{i=1}^{n_e} \|\theta_i\|^2 \leq C \left(\sum_{i=1}^{n_e} \|v_{i0}\|_{H^{k+1}}^2 + \sum_{i=1}^{n_e} \|v_i\|_{L_2(0,T;H^{k+1}(\Omega))}^2 \right)$$

$$+ \sum_{i=1}^{n_e} \left\| \frac{\partial v_i}{\partial t} \right\|_{L_2(0,T;H^{k+1}(\Omega))}^2 \right) h^{2(k+1)}.$$

Hence

$$\|\theta_i\| \leq Ch^{k+1}, \quad i = 1, \dots, n_e$$

and adding the estimate of ρ_i , we obtain the desired result. \square

It is important to note that Gronwall's constant depends on the ratio $\frac{\gamma'_{\max}}{\gamma_0}$. So, if γ'_{\max} is high and γ_0 is small, then, for long time computations, the mesh size should be small enough to compensate for this behaviour.

4 Discrete problem

In this section, we will study the applicability of three known finite difference schemes to discretise in time equation (8). At the end, we will comment the results. Let $\delta > 0$ and consider the partition $]0, T] = \bigcup_{j=1}^{n_i-1}]t_{j-1}, t_j] = \bigcup_{j=1}^{n_i-1} I_j$, $\delta = t_j - t_{j-1}$ and $\text{int}(I_j) \cap \text{int}(I_i) = \emptyset$. Let $\mathbf{V}^{(n)}(y)$ be the approximation of $\mathbf{v}(y, t_n)$ in $(S_h^k)^{n_e}$. In the subsequente, the notation $V^{(n)}$ represents the function V evaluated at time t_n .

4.1 Backward Euler method

First we are going to study the backward Euler method. This method evaluates the equation at the points t_{n+1} , $n = 0, \dots, n_i - 1$, and approximates the time derivative by

$$\frac{\partial \mathbf{V}}{\partial t}(y, t_{n+1}) \approx \frac{\mathbf{V}^{(n+1)}(y) - \mathbf{V}^{(n)}(y)}{\delta} = \bar{\partial} \mathbf{V}^{(n+1)}(y).$$

In this case, system (8) becomes

$$\begin{aligned} \left(\bar{\partial} V_i^{(n+1)}(y), W_i \right) + b_2^{(n+1)} a_i^{(n+1)} \left(\frac{\partial V_i^{(n+1)}}{\partial y}, \frac{\partial W_i}{\partial y} \right) - \left(b_1^{(n+1)} \frac{\partial V_i^{(n+1)}}{\partial y}, W_i \right) \\ = \left(g_i^{(n+1)}, W_i \right), \end{aligned} \quad (9)$$

with $a_i^{(n+1)} = a_i(l^{(n+1)}(V_1^{(n+1)}), \dots, l^{(n+1)}(V_{n_e}^{(n+1)}))$. Recalling the basis $\{\varphi_j\}_{j=1}^{n_p}$, system (9) is a nonlinear algebraic system of the form

$$(\mathcal{M} + \delta \mathcal{A}a(\mathcal{V}^{(n+1)}) - \delta \mathcal{B})\mathcal{V}^{(n+1)} = \mathcal{M}\mathcal{V}^{(n)} + \delta \mathcal{G},$$

with the unknown

$$\mathcal{V}^{(n+1)} = (V_{1,1}^{(n+1)}, \dots, V_{1,n_p}^{(n+1)}, \dots, V_{n_e,1}^{(n+1)}, \dots, V_{n_e,n_p}^{(n+1)}).$$

Due to its nonlinearity, we need to prove the existence of a solution.

Theorem 6. For each $n = 0, \dots, n_i - 1$, system (9) has a solution.

Proof. Let $n \geq 0$ be fixed. For each $h, \delta > 0$, we define the continuous mapping $F : S_h^k \rightarrow S_h^k$ by

$$\begin{aligned} (F(V_i^{(n+1)}), W_i) &= (V_i^{(n+1)}, W_i) - (V_i^{(n)}, W_i) + \delta b_2^{(n+1)} a_i^{(n+1)} \left(\frac{\partial V_i^{(n+1)}}{\partial y}, \frac{\partial W_i}{\partial y} \right) \\ &\quad - \delta \left(b_1^{(n+1)} \frac{\partial V_i^{(n+1)}}{\partial y}, W_i \right) - \delta (g_i^{(n+1)}, W_i). \end{aligned}$$

If $W_i = V_i^{(n+1)}$, then

$$\begin{aligned} (F(V_i^{(n+1)}), V_i^{(n+1)}) &= (V_i^{(n+1)}, V_i^{(n+1)}) - (V_i^{(n)}, V_i^{(n+1)}) - \delta (g_i^{(n+1)}, V_i^{(n+1)}) \\ &\quad - \delta \left(b_1^{(n+1)} \frac{\partial V_i^{(n+1)}}{\partial y}, V_i^{(n+1)} \right) + \delta b_2^{(n+1)} a_i^{(n+1)} \left(\frac{\partial V_i^{(n+1)}}{\partial y}, \frac{\partial V_i^{(n+1)}}{\partial y} \right) \\ &\geq \|V_i^{(n+1)}\|^2 + \delta \frac{\gamma'_{\max}}{2\gamma_0} \|V_i^{(n+1)}\|^2 - \delta \|g_i^{(n+1)}\| \|V_i^{(n+1)}\| + \delta \frac{CM_a}{\gamma_0^2} \|V_i^{(n+1)}\|^2 \\ &\quad - \|V_i^{(n)}\| \|V_i^{(n+1)}\| \\ &= \|V_i^{(n+1)}\| \left(\|V_i^{(n+1)}\| + \delta \frac{\gamma'_{\max}}{2\gamma_0} \|V_i^{(n+1)}\| - \delta \|g_i^{(n+1)}\| + \delta \frac{CM_a}{\gamma_0^2} \|V_i^{(n+1)}\| \right. \\ &\quad \left. - \|V_i^{(n)}\| \right) \\ &= \|V_i^{(n+1)}\| \left(\left(1 + \delta \frac{\gamma'_{\max}}{2\gamma_0} + \delta \frac{CM_a}{\gamma_0^2} \right) \|V_i^{(n+1)}\| - \delta \|g_i^{(n+1)}\| - \|V_i^{(n)}\| \right) \end{aligned}$$

Let us define

$$\varepsilon > \frac{\delta \|g_i^{(n+1)}\| + \|V_i^{(n)}\|}{1 + \delta \frac{\gamma'_{\max}}{2\gamma_0} + \delta \frac{CM_a}{\gamma_0^2}}$$

and

$$B_\varepsilon = \{W \in S_h^k : \|W\| \leq \varepsilon\}.$$

Since $(F(V), V) > 0$ for every $V \in \partial B_\varepsilon$ the corollary to the Brower's Fixed Point Theorem implies the existence of a solution to problem (9). \square

The stability of this method is proved under a condition on the time step.

Theorem 7. Let $\mathbf{V}^{(n+1)}(y)$ be the solution of equation (9). If

$$\delta < \frac{\gamma_0}{\gamma_0 + \gamma'_{\max}}, \quad (10)$$

then

$$\|V_i^{(n+1)}\|^2 \leq C^{n+1} \|V_i^{(0)}\|^2 + \sum_{l=0}^{n+1} C^{n-l+2} \delta \|g_i^{(l)}\|^2,$$

where C could depend on γ'_{\max} , γ_0 and δ .

Proof. Setting $W_i = V_i^{n+1}$ in (9), we obtain the equality

$$\begin{aligned} \left(V_i^{(n+1)}, V_i^{(n+1)} \right) + \delta b_2^{(n+1)} a_i^{(n+1)} \left(\frac{\partial V_i^{(n+1)}}{\partial y}, \frac{\partial V_i^{(n+1)}}{\partial y} \right) &= \left(V_i^{(n)}, V_i^{(n+1)} \right) \\ &\quad - \frac{\delta(\gamma')^{(n+1)}}{2(\gamma)^{(n+1)}} \left(V_i^{(n+1)}, V_i^{(n+1)} \right) + \delta \left(g_i^{(n+1)}, V_i^{(n+1)} \right). \end{aligned}$$

Thus

$$\begin{aligned} \left\| V_i^{(n+1)} \right\|^2 &\leq \frac{1}{2} \left\| V_i^{(n+1)} \right\|^2 + \frac{1}{2} \left\| V_i^{(n)} \right\|^2 + \frac{\delta \gamma'_{\max}}{2\gamma_0} \left\| V_i^{(n+1)} \right\|^2 \\ &\quad + \frac{\delta}{2} \left\| g_i^{(n+1)} \right\|^2 + \frac{\delta}{2} \left\| V_i^{(n+1)} \right\|^2. \\ \left(\frac{1}{2} - \frac{\delta \gamma'_{\max}}{2\gamma_0} - \frac{\delta}{2} \right) \left\| V_i^{(n+1)} \right\|^2 &\leq \frac{1}{2} \left\| V_i^{(n)} \right\|^2 + \frac{\delta}{2} \left\| g_i^{(n+1)} \right\|^2 \end{aligned}$$

From (10), it now follows that

$$\left\| V_i^{(n+1)} \right\|^2 \leq C \left\| V_i^{(n)} \right\|^2 + \delta C \left\| g_i^{(n+1)} \right\|^2,$$

with $C = C(\gamma_0, \gamma'_{\max}, \delta)$. Iterating the result follows. \square

As we suspected, the stability of this method depends on δ and it could be affected if δ is not sufficiently small to compensate for the ratio $\frac{\gamma'_{\max}}{\gamma_0}$. The uniqueness of the solution is proved in the next theorem.

Theorem 8. *If $\delta \approx h^2$ is sufficiently small, then the solution of equation (9) is unique.*

Proof. Suppose that equation (9) has two distinct solutions \mathbf{X} and \mathbf{Y} , then

$$\begin{aligned} (\bar{\partial} X_i, W_i) + b_2^{(n+1)} a_i(l^{(n+1)}(X_1), \dots, l^{(n+1)}(X_{n_e})) \left(\frac{\partial X_i}{\partial y}, \frac{\partial W_i}{\partial y} \right) \\ - \left(b_1^{(n+1)} \frac{\partial X_i}{\partial y}, W_i \right) = (g_i^{(n+1)}, W_i) \end{aligned}$$

and

$$\begin{aligned} (\bar{\partial} Y_i, W_i) + b_2^{(n+1)} a_i(l^{(n+1)}(Y_1), \dots, l^{(n+1)}(Y_{n_e})) \left(\frac{\partial Y_i}{\partial y}, \frac{\partial W_i}{\partial y} \right) \\ - \left(b_1^{(n+1)} \frac{\partial Y_i}{\partial y}, W_i \right) = (g_i^{(n+1)}, W_i) \end{aligned}$$

Subtracting, we arrive at

$$(Y_i - X_i, W_i) + \delta b_2^{(n+1)} \left(a_{i,1}^{(n+1)} \frac{\partial X_i}{\partial y} - a_{i,2}^{(n+1)} \frac{\partial Y_i}{\partial y}, \frac{\partial W_i}{\partial y} \right)$$

$$-\delta \left(b_1^{(n+1)} \left(\frac{\partial X_i}{\partial y} - \frac{\partial Y_i}{\partial y} \right), W_i \right) = 0.$$

Defining $E_i = Y_i - X_i$, it follows that

$$\begin{aligned} (E_i, W_i) + \delta b_2^{(n+1)} a_{i,2}^{(n+1)} \left(\frac{\partial E_i}{\partial y}, \frac{\partial W_i}{\partial y} \right) &= \delta \left(b_1^{(n+1)} \frac{\partial E_i}{\partial y}, W_i \right) + \delta (a_{i,2}^{(n+1)} \\ &\quad - a_{i,1}^{(n+1)}) \left(\frac{\partial X_i}{\partial y}, \frac{\partial W_i}{\partial y} \right). \end{aligned}$$

Setting $W_i = E_i$, we obtain

$$\|E_i\|^2 + \frac{\delta m_a}{\gamma_1^2} \left\| \frac{\partial E_i}{\partial y} \right\|^2 \leq \frac{\delta \gamma'_{\max}}{2\gamma_0} \|E_i\|^2 + \frac{\delta \gamma_1^2 C}{4m_a} \left\| \frac{\partial X_i}{\partial y} \right\|_{\infty} \sum_{j=1}^{n_e} \|E_j\|^2 + \frac{\delta m_a}{\gamma_1^2} \left\| \frac{\partial E_i}{\partial y} \right\|^2,$$

whence

$$\sum_{j=1}^{n_e} \|E_j\|^2 \leq \frac{\delta \gamma'_{\max}}{2\gamma_0} \sum_{j=1}^{n_e} \|E_j\|^2 + \frac{\delta \gamma_1^2 C}{4m_a} \left\| \frac{\partial X_i}{\partial y} \right\|_{\infty} \sum_{j=1}^{n_e} \|E_j\|^2$$

and thus

$$\left(1 - \frac{\delta \gamma'_{\max}}{2\gamma_0} - \frac{\delta \gamma_1^2 C}{4m_a} \left\| \frac{\partial X_i}{\partial y} \right\|_{\infty} \right) \sum_{j=1}^{n_e} \|E_j\|^2 \leq 0.$$

Using the inverse estimates valid in S_h^k , we can prove that

$$\left\| \frac{\partial X_i}{\partial y} \right\|_{\infty} \leq Ch^{-1} \left\| \frac{\partial X_i}{\partial y} \right\| \leq h^{-2} \|X_i\|.$$

By Theorem 7, the result is proved, provided that $\delta \approx h^2$ is sufficiently small. \square

The next theorem establishes optimal convergence order conditions for this scheme.

Theorem 9. *Suppose that δ is small. If \mathbf{v} is the solution of (4) and $\mathbf{V}^{(n+1)}$ is the solution of (9), then*

$$\|V_i^{(n+1)}(y) - v_i(y, t_{n+1})\| \leq C(h^{k+1} + \delta), \quad i = 1, \dots, n_e, \quad n = 1, \dots, n_i,$$

where C does not depend on h , k or δ , but could depend on γ_0 , γ_1 , m_a , γ'_{\max} , α'_{\max} , $\left\| \frac{\partial^2 \mathbf{v}}{\partial t^2} \right\|_{L_{\infty}(0,T;L_2(0,1))}$, $\left\| \frac{\partial \mathbf{v}}{\partial t} \right\|_{L_{\infty}(0,T;H^{k+1}(0,1))}$, $\left\| \frac{\partial \mathbf{v}}{\partial y} \right\|_{L_{\infty}(0,T;L_2(0,1))}$, $\left\| \frac{\partial \mathbf{v}}{\partial y} \right\|_{L_{\infty}(0,T;H^{k+1}(0,1))}$ and $\|\mathbf{v}\|_{L_{\infty}(0,T;H^{k+1}(0,1))}$.

Proof. Set $V_i^{(n+1)} - v_i^{(n+1)} = V_i^{(n+1)} - \tilde{v}_i^{(n+1)} + \tilde{v}_i^{(n+1)} = \theta_i^{(n+1)} + \rho_i^{(n+1)}$. By Lemma 4, we have that

$$\left\| \rho_i^{(n+1)} \right\| \leq Ch^{k+1} \|v_i\|_{H^{k+1}}, \quad n = 1, \dots, n_i.$$

For θ_i , we set

$$\begin{aligned}
& \left(\bar{\partial} \theta_i^{(n+1)}, W_i \right) + b_2^{(n+1)} a_i^{(n+1)} \left(\frac{\partial \theta_i^{(n+1)}}{\partial y}, \frac{\partial W_i}{\partial y} \right) - \left(b_1^{(n+1)} \frac{\partial \theta_i^{(n+1)}}{\partial y}, W_i \right) \\
&= \left(\bar{\partial} V_i^{(n+1)}, W_i \right) + b_2^{(n+1)} a_i^{(n+1)} \left(\frac{\partial V_i^{(n+1)}}{\partial y}, \frac{\partial W_i}{\partial y} \right) - \left(b_1^{(n+1)} \frac{\partial V_i^{(n+1)}}{\partial y}, W_i \right) \\
&\quad - \left(\bar{\partial} \tilde{v}_i^{(n+1)}, W_i \right) - b_2^{(n+1)} a_i^{(n+1)} \left(\frac{\partial \tilde{v}_i^{(n+1)}}{\partial y}, \frac{\partial W_i}{\partial y} \right) + \left(b_1^{(n+1)} \frac{\partial \tilde{v}_i^{(n+1)}}{\partial y}, W_i \right) \\
&= \left(g_i^{(n+1)}, W_i \right) - \left(\bar{\partial} \tilde{v}_i^{(n+1)}, W_i \right) - b_2^{(n+1)} a_i^{(n+1)} \left(\frac{\partial \tilde{v}_i^{(n+1)}}{\partial y}, \frac{\partial W_i}{\partial y} \right) \\
&\quad + \left(b_1^{(n+1)} \frac{\partial \tilde{v}_i^{(n+1)}}{\partial y}, W_i \right) \\
&= \left(\left(\frac{\partial v_i}{\partial t} \right)^{(n+1)}, W_i \right) + b_2^{(n+1)} a_i^{(n+1)} (\mathbf{v}^{(n+1)}) \left(\frac{\partial v_i^{(n+1)}}{\partial y}, \frac{\partial W_i}{\partial y} \right) \\
&\quad - \left(b_1^{(n+1)} \frac{\partial v_i^{(n+1)}}{\partial y}, W_i \right) - \left(\bar{\partial} \tilde{v}_i^{(n+1)}, W_i \right) - b_2^{(n+1)} a_i^{(n+1)} (\mathbf{V}^{(n+1)}) \left(\frac{\partial v_i^{(n+1)}}{\partial y}, \frac{\partial W_i}{\partial y} \right) \\
&\quad + \left(b_1^{(n+1)} \frac{\partial \tilde{v}_i^{(n+1)}}{\partial y}, W_i \right) \\
&= \left(\left(\frac{\partial v_i}{\partial t} \right)^{(n+1)} - \bar{\partial} \tilde{v}_i^{(n+1)}, W_i \right) + b_2^{(n+1)} (a_i^{(n+1)} (\mathbf{v}^{(n+1)}) \\
&\quad - a_i^{(n+1)} (\mathbf{V}^{(n+1)})) \left(\frac{\partial v_i^{(n+1)}}{\partial y}, \frac{\partial W_i}{\partial y} \right) + \left(b_1^{(n+1)} \left(\frac{\partial \tilde{v}_i^{(n+1)}}{\partial y} - \frac{\partial v_i^{(n+1)}}{\partial y} \right), W_i \right).
\end{aligned}$$

Making $W_i = \theta_i^{(n+1)}$ and taking in to account the lower bounds of a and b_2 , we obtain

$$\begin{aligned}
& \frac{1}{2} \bar{\partial} \|\theta_i^{(n+1)}\|^2 + \frac{m_a}{\gamma_1^2} \left\| \frac{\partial \theta_i^{(n+1)}}{\partial y} \right\|^2 \leq \left(b_1^{(n+1)} \frac{\partial \theta_i^{(n+1)}}{\partial y}, \theta_i^{(n+1)} \right) \\
&+ \left(\left(\frac{\partial v_i}{\partial t} \right)^{(n+1)} - \bar{\partial} \tilde{v}_i^{(n+1)}, \theta_i^{(n+1)} \right) + \left(b_1^{(n+1)} \left(\frac{\partial \tilde{v}_i^{(n+1)}}{\partial y} - \frac{\partial v_i^{(n+1)}}{\partial y} \right), \theta_i^{(n+1)} \right) \\
&+ b_2^{(n+1)} (a_i^{(n+1)} (\mathbf{v}^{(n+1)}) - a_i^{(n+1)} (\mathbf{V}^{(n+1)})) \left(\frac{\partial v_i^{(n+1)}}{\partial y}, \frac{\partial \theta_i^{(n+1)}}{\partial y} \right). \quad (11)
\end{aligned}$$

Using the hypothesis (H1) and (H2) and integration by parts, we obtain

$$\left(b_1^{(n+1)} \frac{\partial \theta_i^{(n+1)}}{\partial y}, \theta_i^{(n+1)} \right) = - \frac{(\gamma')^{(n+1)}}{2\gamma^{(n+1)}} \left\| \theta_i^{(n+1)} \right\|^2.$$

By (H5), we have

$$\begin{aligned} |a_i^{(n+1)}(\mathbf{v}^{(n+1)}) - a_i^{(n+1)}(\mathbf{V}^{(n+1)})| &\leq |\gamma^{(n+1)}| \sum_{i=1}^{n_e} C_i \|v_i^{(n+1)} - V_i^{(n+1)}\| \\ &\leq \gamma_1 \left(\sum_{i=1}^{n_e} C_i \|\theta_i^{(n+1)}\| + \sum_{i=1}^{n_e} C_i \|\rho_i^{(n+1)}\| \right). \end{aligned}$$

Taking the absolute value of the expression on the right-hand side of inequality (11) and using the Cauchy inequality, it follows that

$$\begin{aligned} \frac{1}{2} \bar{\partial} \|\theta_i^{(n+1)}\|^2 + \frac{m_a}{\gamma_1^2} \left\| \frac{\partial \theta_i^{(n+1)}}{\partial y} \right\|^2 &\leq \frac{\gamma'_{\max}}{2\gamma_0} \|\theta_i^{(n+1)}\|^2 + \frac{1}{2} \left\| \left(\frac{\partial v_i}{\partial t} \right)^{(n+1)} - \bar{\partial} \tilde{v}_i^{(n+1)} \right\|^2 \\ + \frac{1}{2} \|\theta_i^{(n+1)}\|^2 + \left\| \frac{\partial v_i}{\partial y} \right\|_{L_\infty(0,T;L_2(0,1))} \frac{\gamma_1^2}{4m_a} \gamma_1^2 &\left(\sum_{i=1}^{n_e} C_i \|\theta_i^{(n+1)}\|^2 + \sum_{i=1}^{n_e} C_i \|\rho_i^{(n+1)}\|^2 \right) \\ + \frac{m_a}{\gamma_1^2} \left\| \frac{\partial \theta_i^{(n+1)}}{\partial y} \right\|^2 + \frac{(\alpha'_{\max} + \gamma'_{\max})^2}{2\gamma_0^2} &\left\| \frac{\partial \rho_i^{(n+1)}}{\partial y} \right\|^2 + \frac{1}{2} \|\theta_i^{(n+1)}\|^2. \end{aligned}$$

Interpolation and numerical differentiation theories permit us to prove that

$$\begin{aligned} \left\| \left(\frac{\partial v_i}{\partial t} \right)^{(n+1)} - \bar{\partial} \tilde{v}_i^{(n+1)} \right\|^2 &\leq \left\| \left(\frac{\partial v_i}{\partial t} \right)^{(n+1)} - \bar{\partial} v_i^{(n+1)} \right\|^2 + \left\| \bar{\partial} v_i^{(n+1)} - \bar{\partial} \tilde{v}_i^{(n+1)} \right\|^2 \\ &\leq C\delta^2 \left\| \frac{\partial^2 v_i}{\partial t^2} \right\|_{L_\infty(0,T;L_2(0,1))}^2 + Ch^{2(k+1)} \left\| \frac{\partial v_i}{\partial t} \right\|_{L_\infty(0,T;H^{k+1}(0,1))}^2. \end{aligned}$$

So,

$$\begin{aligned} \bar{\partial} \|\theta_i^{(n+1)}\|^2 &\leq \left(\frac{\gamma'_{\max}}{\gamma_0} + 2 \right) \|\theta_i^{(n+1)}\|^2 + C\delta^2 \left\| \frac{\partial^2 v_i}{\partial t^2} \right\|_{L_\infty(0,T;L_2(0,1))}^2 \\ + Ch^{2(k+1)} \left\| \frac{\partial v_i}{\partial t} \right\|_{L_\infty(0,T;H^{k+1}(0,1))}^2 &+ C \left\| \frac{\partial v_i}{\partial y} \right\|_{L_\infty(0,T;L_2(0,1))} \frac{\gamma_1^4}{2m_a} \sum_{i=1}^{n_e} \|\theta_i^{(n+1)}\|^2 \\ + Ch^{2(k+1)} \left\| \frac{\partial v_i}{\partial y} \right\|_{L_\infty(0,T;L_2(0,1))} \frac{\gamma_1^4}{2m_a} &\|v_i\|_{L_\infty(0,T;H^{k+1}(0,1))}^2 \\ + Ch^{2(k+1)} \frac{(\alpha'_{\max} + \gamma'_{\max})^2}{2\gamma_0^2} &\left\| \frac{\partial v_i}{\partial y} \right\|_{L_\infty(0,T;H^{k+1}(0,1))}^2. \end{aligned}$$

Whence

$$\bar{\delta} \sum_{i=1}^{n_e} \|\theta_i^{(n+1)}\|^2 \leq C_1 \sum_{i=1}^{n_e} \|\theta_i^{(n+1)}\|^2 + C_2(\delta^2 + h^{2(k+1)}),$$

with $C_1 = C_1(\gamma_0, \gamma_1, m_a, \gamma'_{\max}, \left\| \frac{\partial \mathbf{v}}{\partial y} \right\|_{L_\infty(0,T;L_2(0,1))})$ and

$$C_2 = C_2(\gamma_0, \gamma_1, m_a, \gamma'_{\max}, \alpha'_{\max}, \left\| \frac{\partial^2 \mathbf{v}}{\partial t^2} \right\|_{L_\infty(0,T;L_2(0,1))}, \left\| \frac{\partial \mathbf{v}}{\partial t} \right\|_{L_\infty(0,T;H^{k+1}(0,1))},$$

$$\left\| \frac{\partial \mathbf{v}}{\partial y} \right\|_{L_\infty(0,T;L_2(0,1))}, \left\| \frac{\partial \mathbf{v}}{\partial y} \right\|_{L_\infty(0,T;H^{k+1}(0,1))}, \|\mathbf{v}\|_{L_\infty(0,T;H^{k+1}(0,1))}).$$

Hence

$$(1 - \delta C_1) \sum_{i=1}^{n_e} \|\theta_i^{(n+1)}\|^2 \leq \sum_{i=1}^{n_e} \|\theta_i^{(n)}\|^2 + C_2 \delta (\delta^2 + h^{2(k+1)}).$$

If δ is sufficiently small, then, iterating, we obtain

$$\sum_{i=1}^{n_e} \|\theta_i^{(n+1)}\|^2 \leq C_4 \sum_{i=1}^{n_e} \|\theta_i^{(0)}\|^2 + C_3 \delta (\delta^2 + h^{2(k+1)}).$$

The estimates of $\|\theta_i^{(0)}\|$ and $\|\rho_i^{(n)}\|$ complete the proof. \square

Obtaining the solution of (9) implies using an iterative method in each time step. We could apply Newton's method or some secant method, but we choose the fixed point method. For the solution of equation (9), in each time step, we propose the following iterative scheme:

$$(M + \delta A a_i(\mathbf{V}_k^{(n+1)}) - \delta B) V_{i,k+1}^{(n+1)} = M V_i^{(n)} + \delta G_i, \quad i = 1, \dots, n_e, \quad k = 1, 2, \dots \quad (12)$$

with $\mathbf{V}_0^{(n+1)} = \mathbf{V}^{(n)}$ and iterating until $\|\mathbf{V}_{k+1}^{(n+1)} - \mathbf{V}_k^{(n+1)}\| \leq \text{tol}$. Finally we only need to prove that this scheme converges, that is, for a prescribed $\text{tol} > 0$ there exists a $K \in \mathbb{N}$ such that $\|\mathbf{V}_{k+1}^{(n+1)} - \mathbf{V}_k^{(n+1)}\| \leq \text{tol}$ for all $k \geq K$.

Theorem 10. *If δ is sufficiently small, then the iterative scheme (12) converges.*

Proof. The matrices M and A are positive definite, so, if δ is small, then system (12) has a unique solution for any $k = 1, 2, \dots$. Subtracting the systems in two consecutive iterations, say k and $k+1$, we obtain

$$\begin{aligned} & (M + \delta A a_i(\mathbf{V}_k^{(n+1)}) - \delta B)(V_{i,k+1}^{(n+1)} - V_{i,k}^{(n+1)}) \\ &= \delta A \left(a_i(\mathbf{V}_k^{(n+1)}) - a_i(\mathbf{V}_{k-1}^{(n+1)}) \right) V_{i,k}^{(n+1)}, \quad i = 1, \dots, n_e. \end{aligned}$$

Taking the norm on both sides of this equality, and defining $E_{i,k+1} = V_{i,k+1}^{(n+1)} - V_{i,k}^{(n+1)}$, we arrive at

$$\|M + \delta A a_i(\mathbf{V}_k^{(n+1)}) - \delta B\| \|E_{i,k+1}\| \leq \delta C \sum_{j=1}^{n_e} \|E_{j,k}\|.$$

For a small δ , there exists a constant $C_1 > 0$ such that

$$\|M + \delta A a_i(\mathbf{V}_k^{(n+1)}) - \delta B\| \geq C_1, \quad i = 1, \dots, n_e.$$

Summing up for $j = 1, \dots, n_e$, the inequality becomes

$$\sum_{j=1}^{n_e} \|E_{i,k+1}\| \leq \frac{\delta C}{C_1} \sum_{j=1}^{n_e} \|E_{j,k}\|.$$

Iterating,

$$\sum_{j=1}^{n_e} \|E_{i,k+1}\| \leq \left(\frac{\delta C}{C_1}\right)^{k+1} \sum_{j=1}^{n_e} \|E_{j,0}\|.$$

If we choose the time step δ such that $\frac{\delta C}{C_1} < 1$ then, for any $tol > 0$, there exists a K such that for all $k > K$, $\|\mathbf{V}_{k+1}^{(n+1)} - \mathbf{V}_k^{(n+1)}\| \leq tol$. \square

4.2 Crank-Nicolson method

The Crank-Nicolson method evaluates equation (8) at the points $t_{n-1/2} = \frac{t_n + t_{n-1}}{2}$, $n = 1, \dots, n_i$, and uses the approximations

$$\mathbf{V}(y, t_{n-1/2}) \approx \frac{\mathbf{V}^{(n)}(y) + \mathbf{V}^{(n-1)}(y)}{2} = \hat{\mathbf{V}}^{(n)}(y)$$

and

$$\frac{\partial \mathbf{V}}{\partial t}(y, t_{n-1/2}) \approx \frac{\mathbf{V}^{(n)}(y) - \mathbf{V}^{(n-1)}(y)}{\delta} = \bar{\partial} \mathbf{V}^{(n)}(y).$$

Then we have the problem of finding $\mathbf{V}^{(n)} \in (S_h^k)^{n_e}$ such that it is zero on the boundary of Ω , satisfies $V_i^{(0)} = I_h(v_{i0})$, $i = 1, \dots, n_e$, and

$$\begin{aligned} \int_0^1 \bar{\partial} V_i^{(n)} W_i \, dy + a_i(l(\hat{V}_1^{(n)}), \dots, l(\hat{V}_{n_e}^{(n)})) b_2^{(n-1/2)} \int_0^1 \frac{\partial \hat{V}_i^{(n)}}{\partial y} \frac{\partial W_i}{\partial y} \, dy \\ - \int_0^1 b_1^{(n-1/2)} \frac{\partial \hat{V}_i^{(n)}}{\partial y} W_i \, dy = \int_0^1 g_i^{(n-1/2)} W_i \, dy. \end{aligned} \quad (13)$$

System (13) is a non linear algebraic system due to the presence of $a_i(l(\hat{V}_1^{(n)}), \dots, l(\hat{V}_{n_e}^{(n)}))$.

Theorem 11. *For each $n = 0, \dots, n_i - 1$ and $i = 1, \dots, n_e$, system (13) has a solution.*

Proof. The proof is similar to that of Theorem 6. Let $n \geq 0$ be fixed. For each $h, \delta > 0$, we define the continuous mapping $F : S_h^k \rightarrow S_h^k$ by

$$(F(V), W) = (V, W) - (V_0, W) + \frac{\delta}{2} b_2^{(n-1/2)} a_i^{(n-1/2)}(V, V_0) \left(\frac{\partial V}{\partial y}, \frac{\partial W}{\partial y} \right)$$

$$+\frac{\delta}{2}b_2^{(n-\frac{1}{2})}a_i^{(n-\frac{1}{2})}(V, V_0)\left(\frac{\partial V_0}{\partial y}, \frac{\partial W}{\partial y}\right)-\frac{\delta}{2}\left(b_1\frac{\partial V}{\partial y}, \frac{\partial W}{\partial y}\right)-\frac{\delta}{2}\left(b_1\frac{\partial V_0}{\partial y}, \frac{\partial W}{\partial y}\right)-\delta(g, W).$$

If $W = V$ then

$$(F(V), V) = (V, V) - (V_0, V) + \frac{\delta}{2}b_2^{(n-\frac{1}{2})}a_i^{(n-\frac{1}{2})}(V, V_0)\left(\frac{\partial V}{\partial y}, \frac{\partial V}{\partial y}\right) + \frac{\delta}{2}b_2^{(n-\frac{1}{2})}a_i^{(n-\frac{1}{2})}(V, V_0)\left(\frac{\partial V_0}{\partial y}, \frac{\partial V}{\partial y}\right) - \frac{\delta}{2}\left(b_1\frac{\partial V}{\partial y}, \frac{\partial V}{\partial y}\right) - \frac{\delta}{2}\left(b_1\frac{\partial V_0}{\partial y}, \frac{\partial V}{\partial y}\right) - \delta(g, V).$$

Thus

$$\begin{aligned} (F(V), V) &\geq \|V\|^2 - \|V_0\|\|V\| + \frac{\delta m_a C}{2\gamma_1^2}\|V\|^2 + \frac{\delta m_a C}{2\gamma_1^2}\left(\frac{\partial V_0}{\partial y}, \frac{\partial V}{\partial y}\right) + \frac{\delta\gamma'_{\max}}{4\gamma_0}\|V\|^2 \\ &\quad - \frac{\delta}{2}\left\|b_1\frac{\partial V_0}{\partial y}\right\|\left\|\frac{\partial V}{\partial y}\right\| - \delta\|g\|\|V\| \\ &\geq \|V\|^2 - \|V_0\|\|V\| + \frac{\delta m_a C}{2\gamma_1^2}\|V\|^2 - \frac{\delta m_a C}{2\gamma_1^2}\left\|\frac{\partial V_0}{\partial y}\right\|\left\|\frac{\partial V}{\partial y}\right\| + \frac{\delta\gamma'_{\max}}{4\gamma_0}\|V\|^2 \\ &\quad - \frac{\delta}{2}\left\|b_1\frac{\partial V_0}{\partial y}\right\|\left\|\frac{\partial V}{\partial y}\right\| - \delta\|g\|\|V\| \\ &\geq \|V\|^2 - \|V_0\|\|V\| + \frac{\delta m_a C}{2\gamma_1^2}\|V\|^2 - \frac{\delta h^{-1}m_a C}{2\gamma_1^2}\left\|\frac{\partial V_0}{\partial y}\right\|\|V\| + \frac{\delta\gamma'_{\max}}{4\gamma_0}\|V\|^2 \\ &\quad - \frac{\delta h^{-1}}{2}\left\|b_1\frac{\partial V_0}{\partial y}\right\|\|V\| - \delta\|g\|\|V\| \\ &\geq \|V\|(\|V\| - \|V_0\| + \frac{\delta m_a C}{2\gamma_1^2}\|V\| - \frac{\delta h^{-1}m_a C}{2\gamma_1^2}\left\|\frac{\partial V_0}{\partial y}\right\| + \frac{\delta\gamma'_{\max}}{4\gamma_0}\|V\| \\ &\quad - \frac{\delta h^{-1}}{2}\left\|b_1\frac{\partial V_0}{\partial y}\right\| - \delta\|g\|). \\ &\left(1 + \delta\left(\frac{m_a C}{2\gamma_1^2} + \frac{\delta\gamma'_{\max}}{4\gamma_0}\right)\right)\|V\| \geq \|V_0\| + \delta h^{-1}\left(\frac{m_a C}{2\gamma_1^2}\left\|\frac{\partial V_0}{\partial y}\right\| + \frac{1}{2}\left\|b_1\frac{\partial V_0}{\partial y}\right\|\right) \\ &\quad + \delta\|g\| \\ &\Leftrightarrow \|V\| \geq \frac{4\gamma_1^2\gamma_0}{4\gamma_1^2\gamma_0 + 2\delta m_a C\gamma_0 + \delta\gamma_1^2\gamma'_{\max}}\left(\frac{m_a C}{2\gamma_1^2}\left\|\frac{\partial V_0}{\partial y}\right\| + \frac{1}{2}\left\|b_1\frac{\partial V_0}{\partial y}\right\|\right) \\ &\quad + \frac{h}{\delta}\|V_0\| + h\|g\| \end{aligned}$$

Let us define

$$\varepsilon > \frac{4\gamma_1^2\gamma_0}{4\gamma_1^2\gamma_0 + 2\delta m_a C\gamma_0 + \delta\gamma_1^2\gamma'_{\max}}\left(\frac{m_a C}{2\gamma_1^2}\left\|\frac{\partial V_0}{\partial y}\right\| + \frac{1}{2}\left\|b_1\frac{\partial V_0}{\partial y}\right\| + \frac{h}{\delta}\|V_0\| + h\|g\|\right)$$

and

$$B_\varepsilon = \{W \in S_h^k : \|W\| \leq \varepsilon\}.$$

Since $(F(V), V) > 0$, for every $V \in \partial B_\varepsilon$, the corollary to Brower's Fixed Point Theorem implies the existence of a solution to Problem (13). \square

The stability is proved in the next theorem.

Theorem 12. *Suppose that δ satisfies*

$$\delta \leq \frac{4\gamma_0}{\gamma'_{\max} + \gamma_0}. \quad (14)$$

If $\mathbf{V}^{(n)}(y)$ is the solution of (13), then

$$\left\| V_i^{(n)} \right\|^2 \leq C_1 \left\| V_i^{(0)} \right\|^2 + \sum_{l=0}^n C_3^{n-l+2} \delta \left\| g_i^{(l-\frac{1}{2})} \right\|^2,$$

where C_1, C_3 could depend on γ'_{\max}, γ_0 and δ .

Proof. Putting $W_i = \hat{V}_i^{(n)}$ in (13), we obtain

$$\begin{aligned} & (\bar{\partial} V_i^{(n)}, \hat{V}_i^{(n)}) + a_i(l(\hat{V}_1^{(n)}), \dots, l(\hat{V}_{n_e}^{(n)})) b_2^{(n-1/2)} \left(\frac{\partial \hat{V}_i^{(n)}}{\partial y}, \frac{\partial \hat{V}_i^{(n)}}{\partial y} \right) \\ & - \left(b_1^{(n-1/2)} \frac{\partial \hat{V}_i^{(n)}}{\partial y}, \hat{V}_i^{(n)} \right) = \left(g_i^{(n-1/2)}, \hat{V}_i^{(n)} \right). \end{aligned}$$

Since the second term on the left-hand side is non-negative, applying Green's Theorem to the first term on the right-hand side, we obtain

$$\begin{aligned} & \frac{1}{2} \bar{\partial} \|V_i^{(n)}\|^2 \leq \frac{(\gamma')^{(n-\frac{1}{2})}}{2\gamma^{(n-\frac{1}{2})}} \|\hat{V}_i^{(n)}\|^2 + (g^{(n-\frac{1}{2})}, \hat{V}_i^{(n)}) \\ & \leq \frac{\gamma'_{\max}}{8\gamma_0} \left(\|V_i^{(n)}\|^2 + \|V_i^{(n-1)}\|^2 \right) + \frac{1}{2} \|g^{(n-\frac{1}{2})}\|^2 + \frac{1}{8} \left(\|V_i^{(n)}\|^2 + \|V_i^{(n-1)}\|^2 \right). \end{aligned}$$

So,

$$\begin{aligned} \|V_i^{(n)}\|^2 & \leq \|V_i^{(n-1)}\|^2 + \frac{\delta\gamma'_{\max}}{4\gamma_0} \|V_i^{(n)}\|^2 + \frac{\delta\gamma'_{\max}}{4\gamma_0} \|V_i^{(n-1)}\|^2 + \delta \|g^{(n-\frac{1}{2})}\|^2 + \frac{\delta}{4} \|V_i^{(n)}\|^2 \\ & \quad + \frac{\delta}{4} \|V_i^{(n-1)}\|^2. \end{aligned}$$

Collecting the terms, the last inequality becomes

$$\left(1 - \frac{\delta\gamma'_{\max}}{4\gamma_0} - \frac{\delta}{4} \right) \|V_i^{(n)}\|^2 \leq \left(1 + \frac{\delta\gamma'_{\max}}{4\gamma_0} + \frac{\delta}{4} \right) \|V_i^{(n-1)}\|^2 + \delta \|g^{(n-\frac{1}{2})}\|^2.$$

If delta satisfies (14), then

$$\|V_i^{(n)}\|^2 \leq C_1 \|V_i^{(n-1)}\|^2 + C_2 \delta \|g^{(n-\frac{1}{2})}\|^2.$$

Iterating, we obtain the desired estimate. \square

We note that condition (14) permits larger step sizes in time than condition (10).

Theorem 13. *If $\delta \approx h^2$ is sufficiently small, then the solution of (13) is unique.*

Proof. For a fixed n , suppose that $\mathbf{V}^{(n-1)}$ is known and that system (13) has two different solutions, \mathbf{X} and \mathbf{Y} . Subtracting both equations, we obtain

$$\begin{aligned} & (X_i - Y_i, W_i) + \frac{\delta}{2} b_2^{(n-\frac{1}{2})} a_i^{(n-\frac{1}{2})} \left(\frac{\mathbf{X} + \mathbf{V}^{(n-1)}}{2} \right) \left(\frac{\partial X_i}{\partial y}, \frac{\partial W_i}{\partial y} \right) \\ & - \frac{\delta}{2} b_2^{(n-\frac{1}{2})} a_i^{(n-\frac{1}{2})} \left(\frac{\mathbf{X} + \mathbf{V}^{(n-1)}}{2} \right) \left(\frac{\partial X_i}{\partial y}, \frac{\partial W_i}{\partial y} \right) - \frac{\delta}{2} \left(b_1^{(n-\frac{1}{2})} \frac{\partial(X_i - Y_i)}{\partial y}, W_i \right) \\ & + \frac{\delta}{2} b_2^{(n-\frac{1}{2})} \left(a_i^{(n-\frac{1}{2})} \left(\frac{\mathbf{X} + \mathbf{V}^{(n-1)}}{2} \right) - a_i^{(n-\frac{1}{2})} \left(\frac{\mathbf{Y} + \mathbf{V}^{(n-1)}}{2} \right) \right) \\ & \quad \times \left(\frac{\partial V_i^{(n-1)}}{\partial y}, \frac{\partial W_i}{\partial y} \right) = 0. \end{aligned}$$

Defining $\mathbf{E} = \mathbf{X} - \mathbf{Y}$, we can prove that

$$\begin{aligned} & (E_i, W_i) + \frac{\delta}{2} b_2^{(n-\frac{1}{2})} a_i^{(n-\frac{1}{2})} \left(\frac{\mathbf{X} + \mathbf{V}^{(n-1)}}{2} \right) \left(\frac{\partial E_i}{\partial y}, \frac{\partial W_i}{\partial y} \right) - \frac{\delta}{2} \left(b_1^{(n-\frac{1}{2})} \frac{\partial(E_i)}{\partial y}, W_i \right) \\ & + \frac{\delta}{2} b_2^{(n-\frac{1}{2})} \left(a_i^{(n-\frac{1}{2})} \left(\frac{\mathbf{X} + \mathbf{V}^{(n-1)}}{2} \right) - a_i^{(n-\frac{1}{2})} \left(\frac{\mathbf{Y} + \mathbf{V}^{(n-1)}}{2} \right) \right) \\ & \quad \times \left(\frac{\partial(V_i^{(n-1)} + Y_i)}{\partial y}, \frac{\partial W_i}{\partial y} \right) = 0. \end{aligned}$$

Setting $W_i = E_i$ and applying Green's Theorem, we arrive at

$$\begin{aligned} & \|E_i\|^2 + \frac{\delta}{2} b_2^{(n-\frac{1}{2})} a_i^{(n-\frac{1}{2})} \left(\frac{\mathbf{X} + \mathbf{V}^{(n-1)}}{2} \right) \left\| \frac{\partial E_i}{\partial y} \right\|^2 \\ & = \frac{\delta(\gamma')^{(n-\frac{1}{2})}}{4\gamma^{(n-\frac{1}{2})}} \|E_i\|^2 + \frac{\delta}{2} b_2^{(n-\frac{1}{2})} \left(a_i^{(n-\frac{1}{2})} \left(\frac{\mathbf{X} + \mathbf{V}^{(n-1)}}{2} \right) \right. \\ & \quad \left. - a_i^{(n-\frac{1}{2})} \left(\frac{\mathbf{Y} + \mathbf{V}^{(n-1)}}{2} \right) \right) \left(\frac{\partial(V_i^{(n-1)} + Y_i)}{\partial y}, \frac{\partial E_i}{\partial y} \right). \end{aligned}$$

Then

$$\begin{aligned} \|E_i\|^2 + \frac{\delta m_a}{2\gamma_1^2} \left\| \frac{\partial E_i}{\partial y} \right\|^2 & = \frac{\delta \gamma'_{\max}}{4\gamma_0} \|E_i\|^2 + \frac{\delta \gamma_1^2 C}{8m_a} \left\| \frac{\partial(V_i^{(n-1)} + Y_i)}{\partial y} \right\|_{\infty} \sum_{j=1}^{n_e} \|E_j\|^2 \\ & \quad + \frac{\delta m_a}{2\gamma_1^2} \left\| \frac{\partial E_i}{\partial y} \right\|^2, \end{aligned}$$

and so

$$\left(1 - \frac{\delta\gamma'_{\max}}{4\gamma_0} - \frac{\delta\gamma_1^2 C}{8m_a} \left\| \frac{\partial(V_i^{(n-1)} + Y_i)}{\partial y} \right\|_{\infty} \right) \sum_{j=1}^{n_e} \|E_j\|^2 \leq 0.$$

As before, we have

$$\left\| \frac{\partial(V_i^{(n-1)} + Y_i)}{\partial y} \right\|_{\infty} \leq Ch^{-2}(\|V_i^{(n-1)}\| + \|Y_i\|).$$

By Theorem 12, the result is proved, provided that $\delta \approx h^2$ is sufficiently small. \square

Theorem 14. *If v is a solution of equation (4) and V_n is a solution of (13), then*

$$\|V_i^{(n)}(y) - v_i(y, t_n)\| \leq C(h^{k+1} + \delta^2), \quad n = 1, \dots, n_t, \quad i = 1, \dots, n_e,$$

for a certain δ and $C = C\left(M_a, m_a, \gamma_0, \gamma'_{\max}, \alpha'_{\max}, \left\| \frac{\partial v}{\partial y} \right\|_{L_{\infty}(0, T, L_2(\Omega))}, \left\| v \right\|_{L_{\infty}(0, T, H^{k+1}(\Omega))}, \left\| \frac{\partial v}{\partial t} \right\|_{L_{\infty}(0, T, L_2(\Omega))}, \left\| \frac{\partial^2 v}{\partial t^2} \right\|_{L_{\infty}(0, T, L_2(\Omega))}, \left\| \frac{\partial^3 v}{\partial t^3} \right\|_{L_{\infty}(0, T, L_2(\Omega))}, \left\| \frac{\partial^3 v}{\partial y \partial t^2} \right\|_{L_{\infty}(0, T, L_2(\Omega))} \right)$ which doesn't depend on h , k and δ .

Proof. We have

$$\begin{aligned} & (\bar{\partial}\theta_i^{(n)}, W_i) + b_2^{(n-\frac{1}{2})} a_i^{(n-\frac{1}{2})} (\hat{\mathbf{V}}^{(n)}) \left(\frac{\partial \hat{\theta}_i^{(n)}}{\partial y}, \frac{\partial W_i}{\partial y} \right) - \left(b_1^{(n-\frac{1}{2})} \frac{\partial \hat{\theta}_i^{(n)}}{\partial y}, W_i \right) \\ &= (\bar{\partial}V_i^{(n)}, W_i) + b_2^{(n-\frac{1}{2})} a_i^{(n-\frac{1}{2})} (\hat{\mathbf{V}}^{(n)}) \left(\frac{\partial \hat{V}_i^{(n)}}{\partial y}, \frac{\partial W_i}{\partial y} \right) - \left(b_1^{(n-\frac{1}{2})} \frac{\partial \hat{V}_i^{(n)}}{\partial y}, W_i \right) \\ & - (\bar{\partial}\tilde{v}_i^{(n)}, W_i) - b_2^{(n-\frac{1}{2})} a_i^{(n-\frac{1}{2})} (\hat{\mathbf{V}}^{(n)}) \left(\frac{\partial \hat{\tilde{v}}_i^{(n)}}{\partial y}, \frac{\partial W_i}{\partial y} \right) - \left(b_1^{(n-\frac{1}{2})} \frac{\partial \hat{\tilde{v}}_i^{(n)}}{\partial y}, W_i \right) \\ &= (g_i^{(n)}, W_i) - (\bar{\partial}\tilde{v}_i^{(n)}, W_i) - b_2^{(n-\frac{1}{2})} a_i^{(n-\frac{1}{2})} (\hat{\mathbf{V}}^{(n)}) \left(\frac{\partial \hat{\tilde{v}}_i^{(n)}}{\partial y}, \frac{\partial W_i}{\partial y} \right) \\ & \quad + \left(b_1^{(n-\frac{1}{2})} \frac{\partial \hat{\tilde{v}}_i^{(n)}}{\partial y}, W_i \right) \\ &= \left(\left(\frac{\partial v_i}{\partial t} \right)^{(n-\frac{1}{2})}, W_i \right) + b_2^{(n-\frac{1}{2})} a_i^{(n-\frac{1}{2})} (\mathbf{v}^{(n-\frac{1}{2})}) \left(\frac{\partial v_i^{(n-\frac{1}{2})}}{\partial y}, \frac{\partial W_i}{\partial y} \right) \\ & - \left(b_1^{(n-\frac{1}{2})} \frac{\partial v_i^{(n-\frac{1}{2})}}{\partial y}, W_i \right) - (\bar{\partial}\tilde{v}_i^{(n)}, W_i) - b_2^{(n-\frac{1}{2})} a_i^{(n-\frac{1}{2})} (\hat{\mathbf{V}}^{(n)}) \left(\frac{\partial \hat{\tilde{v}}_i^{(n)}}{\partial y}, \frac{\partial W_i}{\partial y} \right) \end{aligned}$$

$$\begin{aligned}
& + \left(b_1^{(n-\frac{1}{2})} \frac{\partial \hat{v}_i^{(n)}}{\partial y}, W_i \right) \\
= & \left(\left(\frac{\partial v_i}{\partial t} \right)^{(n-\frac{1}{2})} - \bar{\partial} \tilde{v}_i^{(n)}, W_i \right) + b_2^{(n-\frac{1}{2})} a_i^{(n-\frac{1}{2})} (\mathbf{v}^{(n-\frac{1}{2})}) \left(\frac{\partial v_i^{(n-\frac{1}{2})}}{\partial y} - \frac{\partial \tilde{v}_i^{(n)}}{\partial y}, \frac{\partial W_i}{\partial y} \right) \\
& + b_2^{(n-\frac{1}{2})} (a_i^{(n-\frac{1}{2})} (\mathbf{v}^{(n-\frac{1}{2})}) - a_i^{(n-\frac{1}{2})} (\hat{\mathbf{V}}^{(n)})) \left(\frac{\partial \hat{v}_i^{(n)}}{\partial y}, \frac{\partial W_i}{\partial y} \right) \\
& - \left(b_1^{(n-\frac{1}{2})} \left(\frac{\partial v_i^{(n-\frac{1}{2})}}{\partial y} - \frac{\partial \hat{v}_i^{(n)}}{\partial y} \right), W_i \right)
\end{aligned}$$

Choosing $W_i = \hat{\theta}_i^{(n)}$, we arrive at

$$\begin{aligned}
& \frac{1}{2} \bar{\partial} \|\theta_i^{(n)}\|^2 + b_2^{(n-\frac{1}{2})} a_i^{(n-\frac{1}{2})} (\hat{\mathbf{V}}^{(n)}) \left\| \frac{\partial \hat{\theta}_i^{(n)}}{\partial y} \right\|^2 = \left(b_1^{(n-\frac{1}{2})} \frac{\partial \hat{\theta}_i^{(n)}}{\partial y}, \hat{\theta}_i^{(n)} \right) \\
& + \left(\left(\frac{\partial v_i}{\partial t} \right)^{(n-\frac{1}{2})} - \bar{\partial} \tilde{v}_i^{(n)}, \hat{\theta}_i^{(n)} \right) + b_2^{(n-\frac{1}{2})} a_i^{(n-\frac{1}{2})} (\mathbf{v}^{(n-\frac{1}{2})}) \left(\frac{\partial v_i^{(n-\frac{1}{2})}}{\partial y} - \frac{\partial \tilde{v}_i^{(n)}}{\partial y}, \frac{\partial \hat{\theta}_i^{(n)}}{\partial y} \right) \\
& + b_2^{(n-\frac{1}{2})} (a_i^{(n-\frac{1}{2})} (\mathbf{v}^{(n-\frac{1}{2})}) - a_i^{(n-\frac{1}{2})} (\hat{\mathbf{V}}^{(n)})) \left(\frac{\partial \hat{v}_i^{(n)}}{\partial y}, \frac{\partial \hat{\theta}_i^{(n)}}{\partial y} \right) \\
& - \left(b_1^{(n-\frac{1}{2})} \left(\frac{\partial v_i^{(n-\frac{1}{2})}}{\partial y} - \frac{\partial \hat{v}_i^{(n)}}{\partial y} \right), \hat{\theta}_i^{(n)} \right)
\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}
& \left(b_1^{(n-\frac{1}{2})} \frac{\partial \hat{\theta}_i^{(n)}}{\partial y}, \hat{\theta}_i^{(n)} \right) = - \frac{(\gamma')^{(n-\frac{1}{2})}}{2\gamma^{(n-\frac{1}{2})}} \|\hat{\theta}_i^{(n)}\|^2, \\
& \left(b_1^{(n-\frac{1}{2})} \left(\frac{\partial v_i^{(n-\frac{1}{2})}}{\partial y} - \frac{\partial \hat{v}_i^{(n)}}{\partial y} \right), \hat{\theta}_i^{(n)} \right) = - \frac{(\gamma')^{(n-\frac{1}{2})}}{2\gamma^{(n-\frac{1}{2})}} (v_i^{(n-\frac{1}{2})} - \hat{v}_i^{(n)}, \hat{\theta}_i^{(n)}) \\
& - \left(b_1^{(n-\frac{1}{2})} (v_i^{(n-\frac{1}{2})} - \hat{v}_i^{(n)}), \frac{\partial \hat{\theta}_i^{(n)}}{\partial y} \right).
\end{aligned}$$

Applying the Hölder and Cauchy inequalities, we obtain the inequality

$$\begin{aligned}
& \frac{1}{2} \bar{\partial} \|\theta_i^{(n)}\|^2 + \frac{m_a}{\gamma_0^2} \left\| \frac{\partial \hat{\theta}_i^{(n)}}{\partial y} \right\|^2 \leq \frac{\gamma'_{\max}}{2\gamma_0} \|\hat{\theta}_i^{(n)}\|^2 + C \left(\left\| \left(\frac{\partial v_i}{\partial t} \right)^{(n-\frac{1}{2})} - \bar{\partial} \tilde{v}_i^{(n)} \right\|^2 \right. \\
& \left. + \left\| \frac{\partial v_i^{(n-\frac{1}{2})}}{\partial y} - \frac{\partial \hat{v}_i^{(n)}}{\partial y} \right\|^2 + \|v_i^{(n-\frac{1}{2})} - \hat{v}_i^{(n)}\|^2 + |a_i^{(n-\frac{1}{2})} (\mathbf{v}^{(n-\frac{1}{2})}) - a_i^{(n-\frac{1}{2})} (\hat{\mathbf{V}}^{(n)})|^2 \right)
\end{aligned}$$

$$+ \frac{m_a}{\gamma_0^2} \left\| \frac{\partial \hat{\theta}_i^{(n)}}{\partial y} \right\|^2,$$

where $C = C(M_a, m_a, \gamma_0, \gamma'_{\max}, \alpha'_{\max}, \|\frac{\partial v_i}{\partial y}\|_{L_\infty(0,T,L_2(0,1))})$. Using interpolation and differentiation theory we can establish the following estimates:

$$\begin{aligned} & \left\| \left(\frac{\partial v_i}{\partial t} \right)^{(n-\frac{1}{2})} - \bar{\partial} \tilde{v}_i^{(n)} \right\| \leq \left\| \left(\frac{\partial v_i}{\partial t} \right)^{(n-\frac{1}{2})} - \bar{\partial} v_i^{(n)} \right\| + \left\| \bar{\partial} v_i^{(n)} - \bar{\partial} \tilde{v}_i^{(n)} \right\| \\ & \leq C\delta \left\| \frac{\partial^3 v_i}{\partial t^3} \right\|_{L_\infty(0,T,L_2(0,1))} + Ch^{k+1} \|v_i\|_{L_\infty(0,T,H^{k+1}(0,1))}; \\ & \left\| \frac{\partial v_i^{(n-\frac{1}{2})}}{\partial y} - \frac{\partial \hat{v}_i^{(n)}}{\partial y} \right\| \leq C\delta^2 \left\| \frac{\partial^3 v_i}{\partial y \partial t^2} \right\|_{L_\infty(0,T,L_2(0,1))}; \\ & \|v_i^{(n-\frac{1}{2})} - \hat{v}_i^{(n)}\| \leq \|v_i^{(n-\frac{1}{2})} - \tilde{v}_i^{(n-\frac{1}{2})}\| + \|\tilde{v}_i^{(n-\frac{1}{2})} - \hat{v}_i^{(n)}\| \\ & \leq Ch^{k+1} \|v_i\|_{L_\infty(0,T,H^{k+1}(0,1))} + C\delta^2 \left\| \frac{\partial^2 v_i}{\partial t^2} \right\|_{L_\infty(0,T,L_2(0,1))}; \\ & |a_i^{(n-\frac{1}{2})}(\mathbf{v}^{(n-\frac{1}{2})}) - a_i^{(n-\frac{1}{2})}(\hat{\mathbf{V}}^{(n)})| \\ & \leq \sum_{j=1}^{n_e} C_i \|v_j^{(n-\frac{1}{2})} - \hat{V}_j^{(n)}\| \leq C \sum_{j=1}^{n_e} \|v_j^{(n-\frac{1}{2})} - \hat{v}_j^{(n)}\| + \|\hat{v}_j^{(n)} - \hat{V}_j^{(n)}\| \\ & \leq C \sum_{j=1}^{n_e} \delta^2 \left\| \frac{\partial^2 v_j}{\partial t^2} \right\|_{L_\infty(0,T,L_2(0,1))} + \|\hat{\theta}_j^{(n)}\| + \|\hat{\rho}_j^{(n)}\|. \end{aligned}$$

So,

$$\frac{1}{2} \bar{\partial} \|\theta_i^{(n)}\|^2 \leq \frac{\gamma'_{\max}}{2\gamma_0} \|\hat{\theta}_i^{(n)}\|^2 + C(\delta^2 + h^{k+1})^2 + C_1 \sum_{j=1}^{n_e} \|\hat{\theta}_j^{(n)}\| + \|\hat{\rho}_j^{(n)}\|$$

but, in this inequality, $C = C(T, M_a, m_a, \gamma_0, \gamma'_{\max}, \alpha'_{\max}, \|v_i\|_{L_\infty(0,T,H^{k+1}(0,1))}, \|\frac{\partial v_i}{\partial y}\|_{L_\infty(0,T,L_2(0,1))}, \left\| \frac{\partial^2 v_i}{\partial t^2} \right\|_{L_\infty(0,T,L_2(0,1))}, \left\| \frac{\partial^3 v_i}{\partial y \partial t^2} \right\|_{L_\infty(0,T,L_2(0,1))}, \left\| \frac{\partial^3 v_i}{\partial t^3} \right\|_{L_\infty(0,T,L_2(0,1))})$. Thus

$$\begin{aligned} & \|\theta_i^{(n)}\|^2 \leq C \frac{\delta \gamma'_{\max}}{2\gamma_0} \|\theta_i^{(n)}\|^2 + \left(1 + \frac{\delta \gamma'_{\max}}{2\gamma_0} \right) \|\theta_i^{(n-1)}\|^2 + C\delta(\delta^2 + h^{k+1})^2 \\ & + C\delta \sum_{j=1}^{n_e} \|\theta_j^{(n)}\|^2 + C\delta \sum_{j=1}^{n_e} \|\rho_j^{(n)}\|^2 + C\delta \sum_{j=1}^{n_e} \|\theta_j^{(n-1)}\|^2 + C\delta \sum_{j=1}^{n_e} \|\rho_j^{(n-1)}\|^2. \end{aligned}$$

Summing for $i = 1, \dots, n_e$ and recalling the estimate for ρ , we obtain

$$\begin{aligned} & \left(1 - C\delta - \frac{\delta\gamma'_{\max}}{2\gamma_0}\right) \sum_{j=1}^{n_e} \|\theta_j^{(n)}\|^2 \\ & \leq \left(1 + C\delta + \frac{\delta\gamma'_{\max}}{2\gamma_0}\right) \sum_{j=1}^{n_e} \|\theta_j^{(n-1)}\|^2 + C\delta(\delta^2 + h^{k+1})^2. \end{aligned}$$

If δ satisfies

$$\delta \leq \frac{2\gamma_0}{2C\gamma_0 + \gamma'_{\max}} \quad (15)$$

then

$$\sum_{j=1}^{n_e} \|\theta_j^{(n)}\|^2 \leq C \sum_{j=1}^{n_e} \|\theta_j^{(n-1)}\|^2 + C\delta(\delta^2 + h^{k+1})^2.$$

Iterating, we arrive at

$$\sum_{j=1}^{n_e} \|\theta_j^{(n)}\|^2 \leq C^n \sum_{j=1}^{n_e} \|\theta_j^{(0)}\|^2 + C(\delta^2 + h^{k+1})^2.$$

Since $\|\theta_j^{(0)}\| \leq Ch^{k+1}\|v_{j0}\|_{H^{k+1}(0,1)}$, adding the estimates of ρ_j , the result follows. \square

For the solution of equation (13), in each time step, we propose the following iterative scheme:

$$\begin{aligned} & (M + \delta A a_i (\frac{\mathbf{V}_k^{(n+1)} + \mathbf{V}^{(n)}}{2}) - \delta B) V_{i,k+1}^{(n+1)} \\ & = (M - \delta A a_i (\frac{\mathbf{V}_k^{(n+1)} + \mathbf{V}^{(n)}}{2}) + \delta B) V_i^{(n)} + \delta G_i, \end{aligned} \quad (16)$$

$i = 1, \dots, n_e$, $k = 1, 2, \dots$, with $\mathbf{V}_0^{(n+1)} = \mathbf{V}^{(n)}$ and iterating until $\|\mathbf{V}_{k+1}^{(n+1)} - \mathbf{V}_k^{(n+1)}\| \leq tol$.

Theorem 15. *If δ is sufficiently small then the iterative scheme (16) converges.*

Proof. The matrices M and A are positive definite, so if δ is small then system (16) has a unique solution for any $k = 1, 2, \dots$. Subtracting the systems in two consecutive iterations, say k and $k+1$, taking the norm on both sides and defining $E_{i,k+1} = V_{i,k+1}^{(n+1)} - V_{i,k}^{(n+1)}$, we obtain

$$\|M + \delta A a_i (\frac{\mathbf{V}_k^{(n+1)} + \mathbf{V}^{(n)}}{2}) - \delta B\| \|E_{i,k+1}\| \leq \delta C \sum_{j=1}^{n_e} \|E_{j,k}\|.$$

For a small δ , there exists a constant $C_1 > 0$ such that

$$\|M + \delta A a_i (\frac{\mathbf{V}_k^{(n+1)} + \mathbf{V}^{(n)}}{2}) - \delta B\| \geq C_1, \quad i = 1, \dots, n_e,$$

and summing up for $j = 1, \dots, n_e$, the inequality becomes

$$\sum_{j=1}^{n_e} \|E_{i,k+1}\| \leq \frac{\delta C}{C_1} \sum_{j=1}^{n_e} \|E_{j,k}\|.$$

Iterating, we obtain

$$\sum_{j=1}^{n_e} \|E_{i,k+1}\| \leq \left(\frac{\delta C}{C_1}\right)^{k+1} \sum_{j=1}^{n_e} \|E_{j,0}\|.$$

If we choose the time step δ such that $\frac{\delta C}{C_1} < 1$ then, for any $tol > 0$, there exists a K such that for all $k > K$, $\|\mathbf{V}_{k+1}^{(n+1)} - \mathbf{V}_k^{(n+1)}\| \leq tol$. \square

4.3 Linearised Crank-Nicolson method

In order to avoid the application of an iterative method in each time step, we implement the linearised method suggested in [24], substituting $\hat{V}_i^{(n)}$ with $\bar{V}_i^{(n)} = \frac{3}{2}V_i^{(n-1)} - \frac{1}{2}V_i^{(n-2)}$ in the diffusion coefficient. So, the totally discrete problem, in this case, will be to calculate the functions $\mathbf{V}^{(n)}$, $n \geq 2$, belonging to $(S_h^k)^{n_e}$, which are zero on the boundary of Ω and satisfy

$$\begin{aligned} & \left(\bar{\partial} V_i^{(n)}, W_i \right) + a_i(l(\bar{V}_1^{(n)}), \dots, l(\bar{V}_{n_e}^{(n)})) b_2^{(n-1/2)} \left(\frac{\partial \hat{V}_i^{(n)}}{\partial y}, \frac{\partial W_i}{\partial y} \right) \\ & - \left(b_1^{(n-1/2)} \frac{\partial \hat{V}_i^{(n)}}{\partial y}, W_i \right) = \left(g_i^{(n-1/2)}, W_i \right), \quad n \geq 2, \quad i = 1, \dots, n_e. \end{aligned} \quad (17)$$

In this way, we have a linear multistep method which requires two initial estimates $\mathbf{V}^{(0)}$ and $\mathbf{V}^{(1)}$. The estimate $\mathbf{V}^{(0)}$ is obtained by the initial condition as $V_i^{(0)} = I_h(v_{i0})$. In order to calculate $\mathbf{V}^{(1)}$ with the same accuracy, we follow [24] and use the following predictor-corrector scheme:

$$\begin{aligned} & \left(\frac{V_i^{(1,0)} - V_i^{(0)}}{\delta}, W_i \right) + a_i(l(V_1^{(0)}), \dots, l(V_{n_e}^{(0)})) b_2^{(1/2)} \left(\frac{\partial}{\partial y} \left(\frac{V_i^{(1,0)} + V_i^{(0)}}{2} \right), \frac{\partial W_i}{\partial y} \right) \\ & - \left(b_1^{(1/2)} \frac{\partial}{\partial y} \left(\frac{V_i^{(1,0)} + V_i^{(0)}}{2} \right), W_i \right) = \left(g_i^{(1/2)}, W_i \right), \quad i = 1, \dots, n_e, \quad (18) \\ & \left(\bar{\partial} V_i^{(1)}, W_i \right) + a_i \left(l \left(\frac{V_1^{(1,0)} + V_1^{(0)}}{2} \right), \dots, l \left(\frac{V_{n_e}^{(1,0)} + V_{n_e}^{(0)}}{2} \right) \right) b_2^{(1/2)} \left(\frac{\partial \hat{V}_i^{(1)}}{\partial y}, \frac{\partial W_i}{\partial y} \right) \\ & - \left(b_1^{(1/2)} \frac{\partial \hat{V}_i^{(1)}}{\partial y}, W_i \right) = \left(g_i^{(1/2)}, W_i \right), \quad i = 1, \dots, n_e. \end{aligned} \quad (19)$$

Systems (17)-(19) are all linear and for small values of δ they always have a unique solution. The proof of the stability of the solutions is similar to that of Theorem 12.

Theorem 16. *If \mathbf{v} is the solution of equation (4) and $\mathbf{V}^{(n)}$ is the solution of (17)-(19), then*

$$\|V_i^{(n)}(y) - v_i(y, t_n)\| \leq C(h^{k+1} + \delta^2), \quad n = 1, \dots, n_i, \quad i = 1, \dots, n_e,$$

where C does not depend on h, k or δ , but could depend on $M_a, m_a, \gamma_0, \gamma'_{\max}, \alpha'_{\max}, \left\| \frac{\partial v}{\partial y} \right\|_{L_\infty(0, T, L_2(\Omega))}, \|v\|_{L_\infty(0, T, H^{k+1}(\Omega))}, \left\| \frac{\partial v}{\partial t} \right\|_{L_\infty(0, T, L_2(\Omega))}, \left\| \frac{\partial^2 v}{\partial t^2} \right\|_{L_\infty(0, T, L_2(\Omega))}, \left\| \frac{\partial^3 v}{\partial t^3} \right\|_{L_\infty(0, T, L_2(\Omega))}$ and $\left\| \frac{\partial^3 v}{\partial y \partial t^2} \right\|_{L_\infty(0, T, L_2(\Omega))}$.

Proof. First, we will determine the estimate for $n = 1$. Let $\theta_i^{(1,0)} = V_i^{(1,0)} - \tilde{v}_i^{(1)}$, $\hat{\theta}_i^{(1,0)} = \frac{\theta_i^{(1,0)} + \theta_i^{(0)}}{2}$ and $\bar{\partial}\theta_i^{(1,0)} = \frac{\theta_i^{(1,0)} - \theta_i^{(0)}}{\delta}$. Arguing in the same way as in Theorem 14 and setting $W_i = \hat{\theta}_i^{(1,0)}$ in (18), we have

$$\begin{aligned} & \frac{1}{2} \bar{\partial} \|\theta_i^{(1,0)}\|^2 + \frac{m_a}{\gamma_0^2} \left\| \frac{\partial \hat{\theta}_i^{(1,0)}}{\partial y} \right\|^2 \leq \\ & \leq C \left(\left\| \left(\frac{\partial v_i}{\partial t} \right)^{(1/2)} - \bar{\partial} \tilde{v}_i^{(1)} \right\| + \left\| \frac{\partial v_i^{(1/2)}}{\partial y} - \frac{\partial \hat{v}_i^{(1)}}{\partial y} \right\| + \sum_{j=1}^{n_e} \|v_j^{(1/2)} - V_j^{(0)}\| \right. \\ & \quad \left. + \|\hat{v}_i^{(1)} - v_i^{(1/2)}\| \right) \left\| \frac{\partial \hat{\theta}_i^{(1,0)}}{\partial y} \right\|. \end{aligned}$$

Using Cauchy's inequality, it follows that

$$\begin{aligned} \bar{\partial} \|\theta_i^{(1,0)}\|^2 & \leq C \left(\left\| \left(\frac{\partial v_i}{\partial t} \right)^{(1/2)} - \bar{\partial} \tilde{v}_i^{(1)} \right\| + \left\| \frac{\partial v_i^{(1/2)}}{\partial y} - \frac{\partial \hat{v}_i^{(1)}}{\partial y} \right\| \right. \\ & \quad \left. + \sum_{j=1}^{n_e} \|v_j^{(1/2)} - V_j^{(0)}\| + \|\hat{v}_i^{(1)} - v_i^{(1/2)}\| \right), \end{aligned}$$

with $C = C(M_a, m_a, \gamma_0, \gamma'_{\max}, \alpha'_{\max}, \left\| \frac{\partial v_i}{\partial y} \right\|_{L_\infty(0, T, L_2(0,1))})$. The following estimates are true for every $i \in \{1, \dots, n_e\}$,

$$\begin{aligned} \left\| \left(\frac{\partial v_i}{\partial t} \right)^{(1/2)} - \bar{\partial} \tilde{v}_i^{(1)} \right\| & \leq \left\| \left(\frac{\partial v_i}{\partial t} \right)^{(1/2)} - \bar{\partial} v_i^{(1)} \right\| + \|\bar{\partial} v_i^{(1)} - \bar{\partial} \tilde{v}_i^{(1)}\| \\ & \leq C\delta^2 + Ch^{k+1}, \\ \left\| \frac{\partial v_i^{(1/2)}}{\partial y} - \frac{\partial \hat{v}_i^{(1)}}{\partial y} \right\| & \leq C\delta \int_{t_0}^{t_1} \left\| \frac{\partial^3 v_i}{\partial y \partial t^2} \right\| dt \leq C\delta^2, \\ \|v_i^{(1/2)} - V_i^{(0)}\| & \leq \|v_i^{(1/2)} - v_i^{(0)}\| + \|v_i^{(0)} - V_i^{(0)}\| \leq C\delta + Ch^{k+1}, \end{aligned}$$

and

$$\|\hat{v}_i^{(1)} - v_i^{(1/2)}\| \leq \|\hat{v}_i^{(1)} - \hat{v}_i^{(1/2)}\| + \|\hat{v}_i^{(1/2)} - v_i^{(1/2)}\| \leq C\delta^2 + Ch^{k+1}.$$

Hence

$$\bar{\partial}\|\theta_i^{(1,0)}\|^2 \leq C(h^{k+1} + \delta)^2,$$

and we have the estimate

$$\|\theta_i^{(1,0)}\|^2 \leq \|\theta_i^{(0)}\|^2 + C\delta(h^{k+1} + \delta)^2 \leq C(h^{2(k+1)} + \delta^3), \quad i = 1, \dots, n_e,$$

where $C = C(T, M_a, m_a, \gamma_0, \gamma'_{\max}, \alpha'_{\max}, \|v_i\|_{L_\infty(0,T,H^{k+1}(0,1))}, \|\frac{\partial v_i}{\partial y}\|_{L_\infty(0,T,L_2(0,1))}, \|\frac{\partial^2 v_i}{\partial t^2}\|_{L_\infty(0,T,L_2(0,1))}, \|\frac{\partial^3 v_i}{\partial y \partial t^2}\|_{L_\infty(0,T,L_2(0,1))}, \|\frac{\partial^3 v_i}{\partial t^3}\|_{L_\infty(0,T,L_2(0,1))})$.

Repeating this process for equation (19), we arrive at

$$\begin{aligned} & \frac{1}{2}\bar{\partial}\|\theta_i^{(1)}\|^2 + \frac{m_a}{\gamma_0^2} \left\| \frac{\partial \hat{\theta}_i^{(1)}}{\partial y} \right\|^2 \\ & \leq C \left(\left\| \left(\frac{\partial v_i}{\partial t} \right)^{(1/2)} - \bar{\partial} \tilde{v}_i^{(1)} \right\| + \left\| \frac{\partial v_i^{(1/2)}}{\partial y} - \frac{\partial \hat{v}_i^{(1)}}{\partial y} \right\| \right. \\ & \quad \left. + \sum_{j=1}^{n_e} \left\| v_j^{(1/2)} - \frac{V_j^{(1,0)} - V_j^{(0)}}{2} \right\| + \|\hat{v}_i^{(1)} - v_i^{(1/2)}\| \right) \left\| \frac{\partial \hat{\theta}_i^{(1)}}{\partial y} \right\|. \end{aligned}$$

In this case, we use the estimate

$$\begin{aligned} \left\| v_i^{(1/2)} - \frac{V_i^{(1,0)} - V_i^{(0)}}{2} \right\| & \leq \|v_i^{(1/2)} - \hat{v}_i^{(1)}\| + \|\hat{v}_i^{(1)} - \frac{V_i^{(1,0)} - V_i^{(0)}}{2}\| \\ & \leq \|v_i^{(1/2)} - \hat{v}_i^{(1)}\| + \frac{1}{2}\|\theta_i^{(1,0)}\| + \frac{1}{2}\|\theta_i^{(0)}\| \\ & \leq C(h^{k+1} + \delta^2) + Ch^{k+1} + C(h^{k+1} + \delta^{\frac{3}{2}}) \\ & \leq C(h^{k+1} + \delta^{\frac{3}{2}}), \end{aligned}$$

and then, by Cauchy's inequality, we conclude that

$$\bar{\partial}\|\theta_i^{(1)}\|^2 \leq C(h^{2(k+1)} + \delta^3),$$

whence

$$\|\theta_i^{(1)}\|^2 \leq \|\theta_i^{(0)}\|^2 + C\delta(h^{2(k+1)} + \delta^3) \leq C(h^{2(k+1)} + \delta^4).$$

To conclude the proof, we obtain the result for $n \geq 2$, applying the same process to equation (17). In this way, we obtain

$$\begin{aligned}
& \frac{1}{2} \bar{\partial} \|\theta_i^{(n)}\|^2 + \frac{m_a}{\gamma_0^2} \left\| \frac{\partial \hat{\theta}_i^{(n)}}{\partial y} \right\|^2 \\
& \leq C \left(\left\| \left(\frac{\partial v_i}{\partial t} \right)^{(n-1/2)} - \bar{\partial} \tilde{v}_i^{(n)} \right\| + \left\| \frac{\partial v_i^{(n-1/2)}}{\partial y} - \frac{\partial \hat{v}_i^{(n)}}{\partial y} \right\| \right. \\
& \quad \left. + \sum_{j=1}^{n_e} \left\| v_j^{(n-1/2)} - \bar{V}_j^{(n)} \right\| + \left\| \hat{v}_i^{(n)} - v_i^{(n-1/2)} \right\| \right) \left\| \frac{\partial \hat{\theta}_i^{(n)}}{\partial y} \right\|.
\end{aligned}$$

Now, we need the estimate

$$\begin{aligned}
\left\| v_i^{(n-1/2)} - \bar{V}_i^{(n)} \right\| & \leq \left\| v_i^{(n-1/2)} - \bar{v}_i^{(n)} \right\| + \left\| \bar{v}_i^{(n)} - \bar{V}_i^{(n)} \right\| \\
& \leq \left\| v_i^{(n-1/2)} - \bar{v}_i^{(n)} \right\| + \left\| \bar{p}_i^{(n)} \right\| + \left\| \bar{\theta}_i^{(n)} \right\| \\
& \leq C\delta^2 + Ch^{k+1} + C(\|\theta_{n-1}\| + \|\theta_{n-2}\|)
\end{aligned}$$

to prove that

$$\bar{\partial} \|\theta_i^{(n)}\|^2 \leq C \sum_{j=1}^{n_e} \|\theta_j^{(n-1)}\|^2 + C \sum_{j=1}^{n_e} \|\theta_j^{(n-2)}\|^2 + C(h^{k+1} + \delta^2)^2, \quad i = 1, \dots, n_e.$$

Summing up for all i , it follows that

$$\bar{\partial} \sum_{i=1}^{n_e} \|\theta_i^{(n)}\|^2 \leq C \sum_{j=1}^{n_e} \|\theta_j^{(n-1)}\|^2 + C \sum_{j=1}^{n_e} \|\theta_j^{(n-2)}\|^2 + C(h^{k+1} + \delta^2)^2.$$

Iterating, we obtain

$$\begin{aligned}
\sum_{i=1}^{n_e} \|\theta_i^{(n)}\|^2 & \leq (1 + C\delta) \sum_{i=1}^{n_e} \|\theta_i^{(n-1)}\|^2 + C\delta \sum_{i=1}^{n_e} \|\theta_i^{(n-2)}\|^2 + C\delta(h^{k+1} + \delta^2)^2 \\
& \leq C \sum_{i=1}^{n_e} \|\theta_i^{(1)}\|^2 + C \sum_{i=1}^{n_e} \delta \|\theta_i^{(0)}\|^2 + C\delta(h^{k+1} + \delta^2)^2
\end{aligned}$$

and, recalling the estimates for $\|\theta_i^{(0)}\|$, $\|\theta_i^{(1)}\|$ and $\|\rho_i^{(n)}\|$, the proof is complete. \square

The conditions on h , δ , γ'_{\max} and γ_0 are the same as those in Theorem 14.

5 Example

The final step is to implement this method using a programming language. To perform this task, we choose the Matlab environment. In this section, we present one example to illustrate the applicability and robustness of the methods, comparing the results with the theoretical results proved and with the

results obtained with the method presented in [21]. We simulate a problem with a known exact solution, which will permit us to calculate the error and confirm numerically the theoretical convergence rates. Let us consider Problem (3) with two equations in Q_t and $T = 1$. The diffusion coefficients are

$$a_1(r, s) = 2 - \frac{1}{1+r^2} + \frac{1}{1+s^2}, \quad a_2(r, s) = 3 + \frac{2}{1+r^2} - \frac{1}{1+s^2},$$

the movement of the boundaries is given by the functions

$$\alpha(t) = -\frac{t}{1+t}, \quad \beta(t) = 1 + \frac{2t}{1+t},$$

the functions $f_1(x, t)$, $f_2(x, t)$, $u_{10}(x, t)$ and $u_{20}(x, t)$ are chosen such that

$$u_1(x, t) = \frac{1}{t+1} \left(\frac{611}{70}z - \frac{10513}{210}z^2 + \frac{646}{7}z^3 - \frac{1070}{21}z^4 \right)$$

and

$$u_2(x, t) = e^{-t} \left(\frac{2047}{140}z - \frac{27701}{420}z^2 + \frac{691}{7}z^3 - \frac{995}{21}z^4 \right)$$

with exact solutions

$$z = \frac{(2t+1)(x+tx+t)}{5t^2+5t+1}.$$

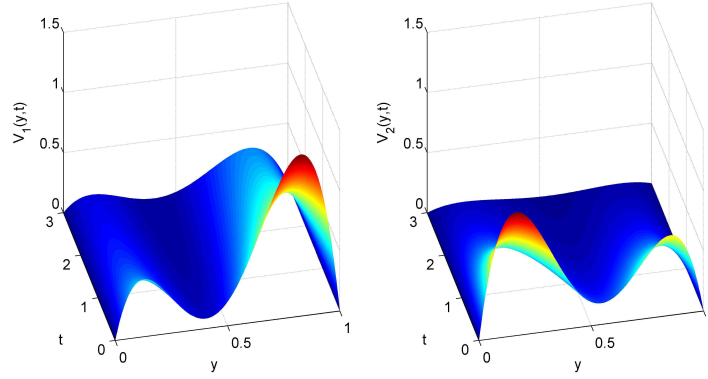


Figure 1: Evolution in time of the approximate solution in the fixed boundary problem for v_1 (left) and v_2 (right).

The picture on the left in Figure 1 illustrates the evolution in time of the solution obtained for v_1 in the fixed boundary problem, and the picture on the right illustrates the evolution in time of the solution obtained for v_2 . This solution was calculated with the linearised Crank-Nicolson method with approximations of degree two and $h = \delta = 10^{-2}$. The pictures in Figure 2 represent

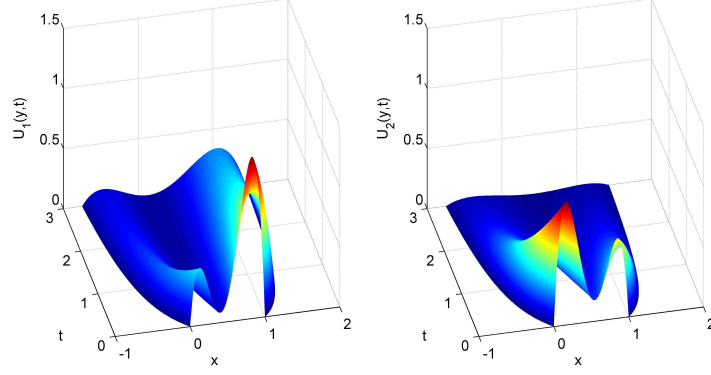


Figure 2: Evolution in time of the approximate solution in the moving boundary problem for u_1 (left) and u_2 (right).

the solutions obtained in the moving boundary domain, after applying the inverse transformation $\tau^{-1}(y, t)$. If, for example, u and v represent the density of two populations of bacteria, we observe that, initially, each population is concentrated mainly in two regions and, as time increases, the two populations decrease and spread out in the domain, as expected.

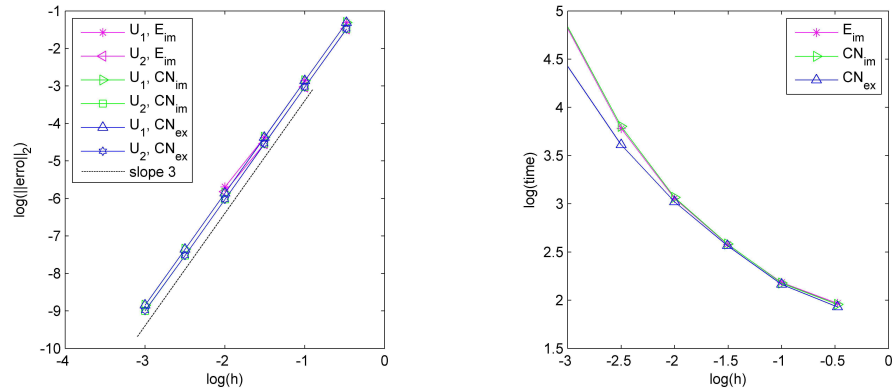


Figure 3: Study of the convergence for h with approximations of degree 2.

In order to analyse the convergence rates, this problem was simulated with different combinations of k , h and δ for each method and the error was calculated at $t = T$ and using the $L_2(\alpha(T), \beta(T))$ -norm in the space variable. In the picture on the left in Figure 3, the logarithms of the errors versus the logarithm of h for the simulations with $\delta = 10^{-4}$ and approximations of degree 2, are represented. As expected, the order of convergence is approximately 3, as was

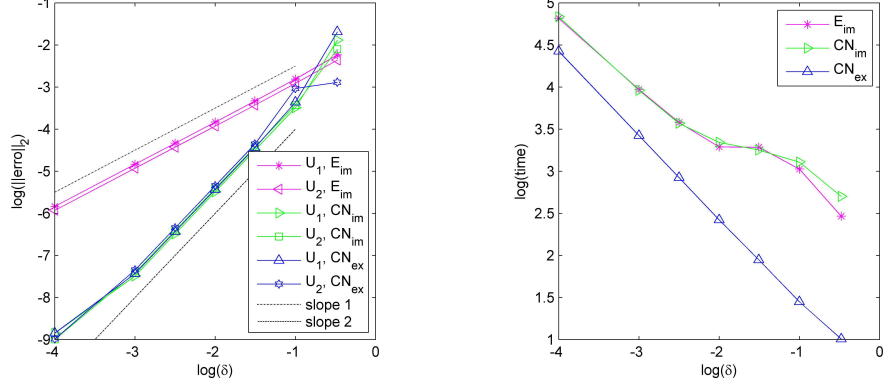


Figure 4: Study of the convergence for δ .

proved in Theorem 5. In the picture on the right we plotted the logarithms of the CPU time versus the logarithm of h . For large h , the three methods took approximately the same time, but as h decreases the implicit methods take more time than the explicit one.

The logarithms of the errors versus the logarithm of δ for the simulations with $h = 10^{-3}$ and approximations of degree 2, are represented in the picture on the left in Figure 4. The results are in accordance with the orders of convergence proved in Theorems 9, 14 and 16.

The logarithms of the CPU time versus the logarithm of δ are plotted in the picture on the right. The implicit methods take much more time than the explicit one for big values of δ , because the fixed point method requires a considerable number of iterations to obtain the predefined tolerance.

In Table 1, we compare the error of the present method with the error of the moving finite element method presented in [21]. All the simulations were done with approximations of degree five and four finite elements. We used $\delta = 10^{-4}$ for the present methods and 10^{-10} for the integrator's error tolerance in the moving finite element method.

6 Conclusions

We established sufficient conditions on the data to obtain optimal convergence rates for some finite element solutions with piecewise polynomial of arbitrary degree basis functions in space when applied to a system of nonlocal parabolic equations. Some numerical experiments were presented, considering different time integrators. The numerical results are in accordance with the theoretical results and are similar in accuracy to results obtained by other method.

	$\max_{j=1,\dots,n_p} \{ u_1(P_j, t_i) - U_1^{(i)}(P_j) \}$			
t_i	MFEM[21]	E_{im}	CN_{im}	CN_{ex}
0.001	7.30e-08	1.26e-07	5.17e-10	2.65e-10
0.005	8.95e-08	5.25e-07	1.56e-09	1.03e-09
0.01	2.79e-08	8.65e-07	2.14e-09	1.46e-09
0.02	1.33e-08	1.33e-06	2.59e-09	1.84e-09
0.05	7.27e-08	2.09e-06	2.73e-09	2.05e-09
0.5	1.90e-08	2.49e-06	1.04e-09	1.06e-09
1	2.12e-08	1.51e-06	4.43e-10	5.06e-10
	$\max_{j=1,\dots,n_p} \{ u_2(P_j, t_i) - U_2^{(i)}(P_j) \}$			
t_i	MFEM[21]	E_{im}	CN_{im}	CN_{ex}
0.001	4.25e-08	4.7e-08	2.26e-10	6.36e-10
0.005	5.20e-08	1.94e-07	5.78e-10	1.45e-09
0.01	1.62e-08	3.25e-07	7.58e-10	1.80e-09
0.02	7.74e-09	5.20e-07	9.01e-10	2.02e-09
0.05	4.22e-08	8.84e-07	1.05e-09	2.16e-09
0.5	1.07e-08	1.45e-06	8.00e-10	1.09e-09
1	9.33e-09	1.13e-06	4.84e-10	5.59e-10

Table 1: Comparison of the present method with the moving finite element method in [21]

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